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KEY

TO

OLNEY'S ELEMENTS OF GEOMETRY

AND

PLANE AND SPHERICAL TRIGONOMETRY,

WITH

AN INTRODUCTION ON METHOD OF TEACHING.

SOLUTIONS BY

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UNIVERSITY OF MICHIGAN.

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1880.

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TEACHING GEOMETRY.

As the text-book to which this volume is a key differs largely in its spirit and plan from the older geometries, it has been thought well that the author should offer a few suggestions as to the best method of using it. In doing this, he trusts that long experience and observation as to methods and results will be deemed a sufficient apology for the broader scope which these suggestions will be seen to take; i. e., suggestions as to methods of teaching the subject, irrespective of the particular text-book used. By permission, the author refers to the articles on Pure Mathematics, and especially to the article on Geometry, in that teacher's thesaurus, "The Cyclopædia of Education," by Kiddle and Schem, as setting forth — more at length than can be done in this place — the author's views on these subjects.

As to Part I. of the Geometry.

The first thing that will strike the teacher upon opening this Geometry is the introductory portion of the work, called *Part First*. This part presents the principal definitions and facts of Plane Geometry, with ample illustrations of their meaning and practical nature. There is no attempt at demonstration. This course has been taken in obedience to the canon of the teacher's art which prescribes "facts before theories," and that the perceptive faculties must supply the facts upon which the logical faculty reasons. A large part of all the difficulty which pupils encounter in the study of geometry grows out of a want of a clear and abiding impression of the subjects of thought and the aims of the science. Geometry is a double science; on one hand it is pure logic, and on the other it is a system of practical truths concerning the properties of form. One may understand thoroughly the logic which demonstrates that the tangent is a mean between the secant and its external segment, and yet know nothing of the practical significance and utility of the fact. Indeed, he may not even apprehend the statement at all as a practical fact of magnitude and form.

As this is a vital point in teaching geometry, let us illustrate it. Suppose a class of a dozen ordinary pupils have come in the old way to this proposition, and have learned to demonstrate it satisfactorily. Then let the teacher draw a square on the tangent, and ask who can draw an oblong of the same size. Not half — perhaps not one — of the class can do it, and many will not even know what is meant. Allow an illustration from the author's experience. During the past twenty years he has examined scores of young men, — candidates for advanced standing in college, — who had been over geometry in the ordinary way. Whenever he has found one who showed special attainment, he has asked something like the following series of questions: —

How do you find the area of a spherical triangle?

What is a tri-rectangular triangle? What part of the surface of a sphere?

What is spherical excess?

Can you demonstrate the truths declared?

In most cases the questions have been answered promptly and correctly, and one or more demonstrations have been given.

A practical problem has then been given: Here is a sphere two feet in diameter; that spherical triangle on its surface has its angles 95°, 80°, and 110°. What is its area?

No student of all the number has been found who could obtain the answer!

It is such reflections and such facts as these, confirmed by the views of such eminent educators as ex-President Hill of Harvard, and the concurrent sentiment of writers on pedagogy, that have given Part I. its character and form.

That the method of giving some acquaintance with the practical truths of geometry, before attempting to demonstrate them, works well in practice, has been demonstrated again and again. Near twenty years ago, the author was teaching in a school where both sexes were in his classes. Conic sections, taught by the old method, never failed to be a drudgery to the girls, and to all but a few of the boys. At length he hit upon this plan: The first few lessons were occupied in teaching how to draw the conic sections neatly and accurately; how to draw a tangent to each; how to find the centre, axes, foci, &c. In fact, all the leading properties were taught *as facts;* and the pupils were trained to draw, with great neatness and elegance, the figures illustrating or applying them. In this way girls and all became enthusiastic, and "Why is it so?" became an oft-repeated and eager question. After this introduction, the terror and drudgery of the science disappeared. They knew what the truths were, and were interested in them as facts, and had an eager desire to know why they were so.

This, the author believes, will inevitably be the result of teaching geometry in the method proposed; indeed, he knows it may be so, from the large number of testimonies he has received from teachers *who have tried the method in the spirit of it.*

WHAT, THEN, IS THE METHOD?—It is pretty fully indicated in the book itself. It would be well if Part I. were introduced into the course in what is usually called the "grammar grade" of our public schools, as soon as an elementary knowledge of arithmetic is secured, and before that subject is completed, or algebra is commenced.

Pupils of this age memorize readily, and are fond of drawing. "The elements of geometry are much easier to learn, and are of more value when learned, than advanced arithmetic; and, if a boy is to leave school with merely a grammar-school education, he would be better prepared for the active duties of life with a *little* arithmetic and *some* geometry, than with *more* arithmetic and *no* geometry." — MARKS. "It will be found that children of this age are quicker at comprehending first lessons in geometry than those of fifteen. The child's powers of sensation are developed before his powers of conception, and these before his reasoning powers. This is, therefore, the true order of education; and a powerful logical drill, like mental arithmetic, is sadly out of place in the hands of a child whose powers of observation and conception have, as yet, received no training whatever." — EX-PRESIDENT HILL *of Harvard.*

That something of this kind is needed in our lower schools, is apparent in the tendency to introduce lessons on form, drawing, &c. But why not make these lessons such as to lay a good foundation for subsequent study? Let Part I. occupy about half of the third year of this grade, and it will be found to be pursued with delight, and will greatly facilitate the subsequent study of geometry. But, if the study of this part is not introduced here, it should not be omitted or slighted when the study of geometry proper is commenced a few years later. A high-school or preparatory class commencing the study of geometry at the ordinary age will save time, and also secure a more intelligent interest in the subject, by spending six weeks on this part.

The points to be secured are: —

1. To learn, thoroughly and *intelligently*, all the definitions.

These should be committed to memory, and repeated until they can be recalled without conscious effort; great pains being taken that the exact language of the text is used, unless the teacher substitutes other language for reason.

Of course we do not mean that this shall be a mere *memoriter* exercise. The greatest pains should be taken to secure a thorough understanding of the technical terms, and of the language of the definitions. For this purpose the illustrations in the text will afford helps to the pupil and hints to the teacher.

A pupil should always be able, not only to recite the definition readily and accurately, but also to illustrate its meaning in the manner of the book, in his own language, and by a figure, when that is possible.

Clearness of perception, and readiness and accuracy of expression, is the motto in doing this work.

2. The second line of effort should be to train the pupil to solve every problem with neatness and despatch. Much practice in drawing smooth lines *in ink*, with a ruling-pen, will be necessary in order that the pupils shall be able to execute the figures properly.

An excellent plan is to have each pupil make a portfolio of drawings with the utmost care on separate sheets of drawing-paper. This should contain every exercise of Part I. A portfolio of separate sheets is better than a book in which the pages are fastened, since no drawings need be put into it till they are satisfactory. A little care at the outset will make this a most pleasing and profitable course of study.

In this part of the work, never allow a pupil to rest satisfied with being able to tell how a thing is to be done, but require that he actually do it with ruler and dividers. These two instruments, with a ruling-pen, are the sole ones needed, or which should be allowed in this work.

When working in class, on the blackboard, a string will take the place of dividers. Never allow a pupil to guess at a construction, at this stage of his course. If a line is said to be perpendicular to another, let it be made so on geometrical principles. If an angle is said to be bisected, let the geometrical construction be made.

This *punctilio* in doing all on correct principles will secure several very important results: 1st, It will fix permanently in mind all the working facts of plane geometry. 2d, It will train the eye to the perception of form. After a course of training like this, a pupil when making a freehand drawing (as is sometimes allowable and even desirable), in subsequent work, will not make an angle of 80° for a right angle, or draw any sort of a triangle when his problem requires an equilateral triangle. His trained geometric sense will not allow this. 3d, This habit of *seeing* the import of geometrical language will give a clearness of apprehension, and a knowledge of relations, which will be of vast importance in subsequent work. 4th, It will give him the *geometrical spirit*, i.e., the instinct, if we may so call it, that leads him to investigate by constructions, and by scrutinizing relations as they appear in geometrical figures.

As to Part II.

The first thing that will strike one here is the division of the subject into chapters and sections, instead of books, as has been the prevailing custom since the time of Euclid (about 300 or 400 B.C). The author's reasons for this are the same as have led writers on most other subjects to abandon the ancient method of division into books, for the modern one of division into chapters and sections. These are, mainly, two: viz., first, the word "book" has become nearly obsolete in the former sense; and, second, the division into chapters, and the subdivision into sections, furnishes a better opportunity for an analysis of the subject. Elementary geometry is naturally divided by its subject-matter into plane and solid; i. e., into the treatment of forms and magnitudes confined to one plane, and of those not so restricted. We thus need, at the outset, a more comprehensive division than the old "book," in order to mark the natural division of the subject. Again, these parts need subdividing, in order to any proper analysis of the subject-matter. In fact, the old arrangement gave us the anomaly of the most perfect of the sciences presented without any logical analysis of its subject-matter. The great natural division into plane and solid was not indicated in any way. Such later editors as have marked this division have found no term by which to designate the parts. Furthermore, the subject-matter of most of the books was utterly heterogeneous. Observe Book I., of Legendre. What is its subject? No one can tell. Indeed, it has no single theme. It treats of the relative position of two lines, of triangles, of quadrilaterals, and of quite incongruous properties of these forms. On the other hand, this volume claims to present a thoroughly logical and scientific analysis of the subject-matter of geometry. There is no section without a clearly defined subject, and the order of arrangement of sections is as logical as the division of the general subject which gives rise to them.

The analysis of the subject-matter of geometry, in accordance with which this treatise is written, is based upon the following considerations: —

1. The concepts of plane geometry are the *straight line*, the *circumference of the circle*, and the *angle*.

2. The two fundamental inquiries are concerning *magnitude* and *form*, — the latter of which results from position.

The straight line — being the simplest and most fundamental concept — must be examined first. But shall our inquiry be with reference to *magnitude*, or with reference to *position*, — which is the basis of form ? As to magnitude, there are two ways of measuring a line, — the direct and the indirect. The direct measurement is effected by applying one line to another, and is so elementary as not to be the subject of scientific investigation. The indirect, as the determining of one of the sides of a triangle, when three other parts are known, or the circumference of a circle from the radius, though a subject for scientific investigation, is not a fundamental inquiry, but a remote result. Hence our inquiry will not be as to magnitude. But the *relative position* of right lines is an important fundamental inquiry: indeed, it lies at the foundation of all geometrical inquiry. Here, then, we have the first theme, — *The relative position of two straight lines.*

As subdivisions, we shall, of course, have (*a*) *Perpendicularity*, (*b*) *Obliquity*, (*c*) *Parallelism.* Here we have the beginning of a system based upon a logical analysis of the subject-matter.

A similar line of inquiry as to the straight line in connection with the circumference of the circle, leads us to the general theme, *The Relative Position of a Straight Line and a Circumference.* Hence we have the fundamental doctrine of chords, secants, and tangents.

From these considerations we pass in the same line of thought to the section on *The Relative Position of Two Circumferences.*

We thus reach the *angle.* Here the line of inquiry changes. The relative position of angles is not an important theme; but the relative *magnitude* of angles is most important, and is fundamental. Moreover, we do not usually measure angles directly, but indirectly, by means of arcs. Hence we have our next theme, *The Measurement of Angles.*

Having now examined the fundamental concepts, we are prepared to investigate the properties of figures which result from their combination. As the parts of a polygon are the sides and angles, the first inquiry will naturally be concerning these, separately, and in relation to each other. This gives us a section of *The Angles of Polygons*, and the *Relations between the Angles and Sides.*

Passing from the consideration of the parts of a single figure, we come naturally to the comparison of figures. There are three lines of comparison usually embraced in elementary geometry; viz., *Equality*, *Equivalency*, and *Simi-*

larity (to these might be added isoperimetry, maxima and minima, symmetry, &c.). Thus we conclude the analysis of plane geometry, with sections on equality, equivalency, and similarity.

We have here a simple and natural outline of the subject, which should be fixed in the mind of the student as he goes over it. It assigns each proposition to its proper place, for an evident and sufficient reason, and presents a comprehensive view of the whole subject.

HOW TO MAKE REFERENCES.

But does not this breaking-up of the subject into short sections make reference difficult? Yes, if we are to make our references in the meaningless way of other days, by book and number of proposition. But if we make our reference, when demonstrating, by quoting the principle referred to, either in full or by definite allusion, it will facilitate reference. To illustrate: in demonstrating that *triangles which are mutually equiangular are similar* (335), we may say, in the old method, "Now, by Book IV., Prop. 12, CD′E′ and D′E′A are to each other as their bases;" or, by the method proposed, we may say, "Now, since triangles which have equal altitudes are to each other as their bases, we have," &c. This is an example of a full quotation. The text-book affords an example of a specific allusion. The number of the article — in this case (324) — as given in the text is not to be given in oral demonstration. This is simply for the student's use in studying. Such a system of reference as this will inevitably render the truths of geometry, and their statement, perfectly familiar to the student. In short, "Book IV., Prop. 12," may or may not mean something by *association;* but it *says nothing.*

THE NOTATION.

Some have thought that they could prevent memorizing by using figures on the diagrams instead of letters (see next topic). To this there are three very serious objections: first, it is grossly inelegant; second, it is not the notation of the science; and, third, *it does no good.* A student who is so disposed can memorize as well (perhaps better) the form of the figure as he can the letters used. It will be found, that, in the parrot-like learning referred to, it is more generally the form of the figure that is fixed in memory, than the letters by which the parts are designated; and neither the slovenly style, "line this," "line that," &c., sometimes heard, nor the use of figures, will at all tend to break up the evil of memorizing. Therefore use the letters. They are the notation of the science. Indeed, it is legitimate, sometimes to use with rigid accuracy the same letters as are used in the book, although this is not well in the earlier part of the course. But, as we advance into the higher departments of the subject, the notation becomes an essential part of the science, without rigid adherence to which it would be impossible to carry on our investigation. In all cases, a neat, compact, easily read, and expressive notation is a very important part of mathematical science.

EXERCISES.

It is a singular fact, and one by no means creditable to our teaching, that hitherto no text-book on geometry has been popular, in this country, which embraced practical problems as an *essential* part of the treatment. In all our books which have shared largely the popular favor, such problems have been wholly wanting, or the arrangement has been such that they did not form an essential part of the treatment, and hence could be "skipped." It has been a leading purpose of the author in constructing this book, to make the practical exercises as much a part of the treatment, as are the examples or problems in arithmetic and algebra. There is no more reason why the student should study geometry as a discussion of abstract propositions, than there is for his studying arithmetic or algebra in this manner. It would be no more absurd to put into the hands of a pupil, as a text-book on arithmetic, a treatise which gave, in the subject of *fractions*, the definitions, the propositions on which the operations are founded, and their demonstrations, and the rules with their demonstrations, — without *practical examples*, — than to put into his hands a geometry which treated the subject of perpendiculars, parallels, equality, &c., without such practical exercises. He would obtain about as much knowledge of arithmetic from the former, as he would of geometry from the latter.

But there is still another point with reference to these exercises that needs our careful attention. In the study of the elements of the subject, these exercises should be such as will bring out the meaning and practical utility of the abstract proposition discussed, and not added abstract propositions for the pupil to demonstrate for himself. To the latter class of exercises there are two very grave objections: first, the average pupil, at this stage of his course, has no ability to do this kind of work; and, second, it is not the thing he needs if he could do it. What he needs is exercises which will make him feel that geometry is a science which has to do with common affairs; that it is not a system of misty abstractions in an imaginary world. It is not more abstraction that he needs, but the abstract discussions made real by applications to practical problems.

This character of exercises is even more important in the earlier part of the course than in the later. It gives real significance to the abstract and often otherwise meaningless discussions, fixes the principle in mind, and develops a spirit and style of thought which is invaluable in the subsequent course, and which once established will never be lost. Furthermore, experience proves that numerical applications are essential to a full and satisfactory apprehension of geometrical facts. A pupil may be able to find a mean proportional between two lines given geometrically, and be quite at fault if they are given arithmetically, and especially if the problem is a little indirect, as "To represent geometrically the square root of 7."

Now, this is by no means all theory, nor a discovery for which the author claims a copyright. It is the spirit, and, in essential particulars, the method which is universally used in the German schools, and which some of our best teachers of geometry have been using for a generation in this country. What the author hopes to accomplish by constructing a text-book on this plan, is to help to make this system of teaching as generally prevalent in this country as it is in Germany.

METHODS OF CONDUCTING RECITATIONS, OR DRILL-EXERCISES, IN PART II.

The definitions and theorems should be repeated with perfect accuracy until the facts and the words are so firmly fixed in mind that they can be recalled as we recall the alphabet or the multiplication-table. In order to this it will (*a*) be best to have them quoted in full in the demonstration; (*b*) another very useful exercise for this purpose is to spend a part of each recitation with questions like the following: —

An inscribed angle — how measured?

A tangent and secant — relation of?

Arcs intercepted by chords?

Parallels cut by secant — the equal angles? the supplementary angles? &c.

These, it will be seen, are mere hints in the form of questions, which give no clew to the answer. *The response should generally be in the exact terms of the theorem stating the answer.* So also, in assigning subjects for recitation, the teacher should not enunciate the proposition, but by some such half-questions as the above, and the following, suggest the topic: —

The relation between the hypotenuse and the sides of a right-angled triangle?

The relative position of two circles when the distance between the centres is less than the sum and greater than the difference of the radii?

The sum of the angles of a triangle?

The relation between the angles and the sides of a triangle? &c.

In this manner the teacher should always designate the proposition *without stating it.* The statement is one of the most important things for the pupil to learn, and have at perfect command, and hence should not be given him by the teacher.

Demonstrations should not be memorized by the pupil; and considerable latitude may be allowed in the use of language, provided the argument is brought out clearly. But errors in grammar, and inelegancies in style, should be carefully guarded against. One of the chief benefits to be derived from class-room drill in mathematics is the ability to think clearly and logically, and to express the thought in concise, perspicuous, and elegant language.

Generally the topic should be assigned the pupil in the manner indicated above, before he leaves his seat, or while he is standing at the blackboard; after which he should be required to construct the figure necessary for the demonstration without helps from any source.

It is quite desirable that the pupil stand at the board when he demonstrates, and trace the diagram as he goes through with the demonstration. In this way full demonstrations should be given.

The giving of synopses of demonstrations is a more difficult exercise, but is a very important one. A synopsis should simply name in order the conclusions reached by the various steps in the argument, without giving the reasons for them. Thus, suppose the theorem to be (214) of the text-book. After stating the theorem, the pupil will say, "This is proved by drawing a chord from the intersection of one of the secants with the circumference, parallel to the other secant. The angle thus formed is equal to the given angle, and is measured by half an arc which is the difference between the two arcs intercepted by the secants." Advanced pupils should be trained to give, without diagrams, synopses of all the propositions; and the practice should be persisted in until such synopses become perfectly familiar. But *the pupil should never confound* a synopsis with a full demonstration. When he professes to give the latter, his language should be exact, and the argument brought out in every point. This is the drill in logic. The drill on synopses is calculated to fix in mind the steps of the argument so that the demonstration can be reproduced. The synopsis is what one should remember, and not the details of the demonstration. Hence the importance of a great amount of drill in giving synopses. For lack of this, pupils who do very satisfactory work in class lose all ability to reproduce the demonstrations soon after leaving the study.

"Make haste slowly" in going through Part II. *Do not neglect the exercises.* There are not half as many exercises in the text as any thorough teacher will require for the illustration and application of the subject. By this is not meant that, in first going over this part, many *new theorems* will be given for original demonstration; but many exercises illustrating and applying the theorems of the text. The pupil should be put in perfect possession of the essential working principles of the science, (*a*) by such familiarity with theorems, corollaries, &c., as will enable him to give instantly the statement in any case, in response to such allusions as are given above; (*b*) by thorough acquaintance with the demonstrations; (*c*) by synopses repeated till they can be recalled without effort; and (*d*) by such exercises as are described above illustrating every proposition, and giving it a concrete, practical significance.

RECAPITULATION.

Let us in conclusion recapitulate the leading features of the plan.

1. Introduce the definitions and leading facts of plane geometry early in the school course, as early, at least, as the middle of the grammar-school grade.

2. Make no attempt at teaching the systematic logic of geometry, at this stage.

3. Fix firmly in mind the fundamental definitions of the science, *in exact language*, and illustrate them so fully that the terms cannot be used in the hearing of the pupil, or by him, without bringing before his mind, *without conscious effort*, the geometrical conception.

4. By numerous and varied applications of the fundamental principles of plane geometry to the most familiar and homely things in common life, divest the pupil's mind of the impression that he is studying "higher mathematics" (as he is not), and beget in him the habit of seeing the applications and illustraitons of these principles everywhere about him.

5. By means of much experience in the elements of geometrical drawing, train the taste to enjoy, the eye to perceive, and the hand to execute, geometrical forms, and by so doing fix indelibly in the mind the "working facts" of geometry.

6. Passing to Part II., having, in the method described above, familiarized the pupil with the concepts and principal truths of the science, and thus prepared the way for the exercise of the logical faculty upon these notions, enter upon the specific work of training this faculty, or of developing in the pupil's mind the logical aspect of the science,

not, however, forgetting the mechanical aspect, nor failing to keep in mind the fact that the abstract logical conceptions will be greatly helped by clear conceptions of the concrete, or practical applications.

7. Have all definitions, theorems, corollaries, &c., memorized with perfect exactitude, and repeated till they can be given without effort. Demonstrations should not be memorized, but *should* be given with fulness, and logical and grammatical accuracy.

8. The teacher should never give a theorem or corollary in proper form, but should suggest the topic (see above), or state the problem which calls for the theorem.[1]

9. The construction of the figure is a necessary part of the demonstration, and no assistance should be given the pupil, nor aids allowed.

10. All figures in plane geometry should, upon first going over the subject, be constructed by the pupils with strict accuracy, on correct geometrical principles, using ruler and string; and this should be persisted in until it can be done with ease. *In reviews*, free-hand drawing of figures may be allowed, and is even desirable.

11. The ordinary notation by letters should be used.

12. All the exercises in the book should be worked with care in the study, and in the class, and be carefully explained by the pupil; and as many additional, impromptu exercises as may be found necessary in order to render the pupil familiar with the practical import of the propositions.

13. Little, if any, original demonstration of theorems not in the book should be required of the pupil upon first going over plane geometry. In review, more or less of such work may be required.

14. Great pains should be taken that original demonstra-

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[1] Thus, "To ascertain the relative areas of two similar triangles," the pupil will say, "Similar triangles are to each other as the squares described on their homologous sides."

tions be given in good, workmanlike *form*. For this purpose, they should be written out with care by the pupil. Indeed, it is an excellent occasional exercise, to have demonstrations written out in full in class.

15. In review, much attention should be given to synopses of demonstrations. They are the main reliance for fixing in memory the line of argument by which a proposition is demonstrated.

KEY

TO THE

SPECIAL OR ELEMENTARY GEOMETRY.

INTRODUCTION.

[NOTE.—The full-faced figures in connection with the number of the page refer to the Articles in the Geometry. The numbers at the left hand, or above the Example, refer to the particular Example. Figures in the Geometry are denoted FIG. 37, etc.; Figures in the Key, FIG. 37*a*, etc.]

SURFACES.

Page 6, 21

1. An egg.
3. A conical surface.
4. Yes. No. Curved.
5. No. No. Curved.
6. Definition of a Plane Surface.
7. Definitions of Plane and Curved Surfaces.
8. By moving in the direction of its length.

A SOLID.

Page 9, 27.

1. A sphere.
2. A cylinder.
3. A cone.
5. A cylinder. The circumference of the circle. The circle itself.

EXTENSION AND FORM.

Page 10, 31.

1. A point.
2. A line. A point.
3. A surface. A line. A point.
4. A spherical surface.

PART I.

ABOUT STRAIGHT LINES.

Page 12, 33.

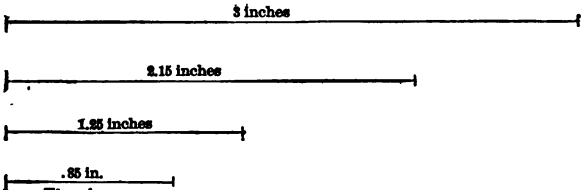

Fig. 1 *a*.

5. See first line, Fig. 1*a*.
6. See second line.
7. See third line.
8. See fourth line.
9. Length, 8.02 inches; width, 5.4; diagonal, 9.65.

Pages 13, 14, 34, 35.

1. See (1.), Fig. 1 *a*.

2 and 3. See (2.) and (3.).

1. See (4.), Fig. 2 *a*.
2. See (5.).
3. See (6.).
4. See (7.).

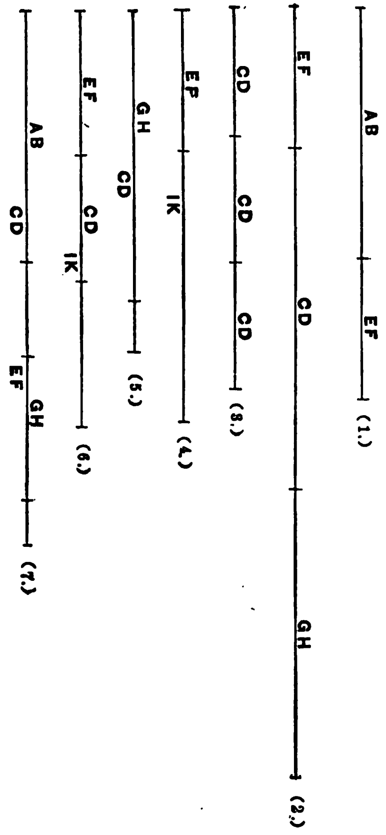

Fig. 2 *a*.

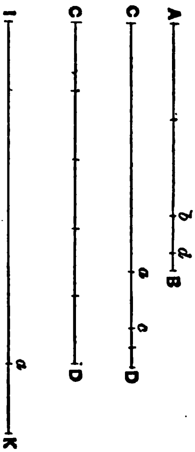

Fig. 3 *a*.

Page 14, 36.

1. Applying **AB** to **CD**, I have a remainder *a***D**; applying *a***D** twice to **AB**, I have a remainder *b***B**; applying this to *a***D**, I have a remainder *c***D**; and, applying this to *b***B**, I have a remainder *d***B**, which is contained in *c***D** twice. By adding the parts it is easily seen that **AB : CD** :: 13 : 18.

2. Applying **CD** to **IK**, I have a remainder *a***K**; applying *a***K** to **CD**, I find it contained five times. ∴ **CD : IK** :: 5 : 6.

3. Applying EF to GH, I find it contained twice. ∴ EF : GH :: 1 : 2.

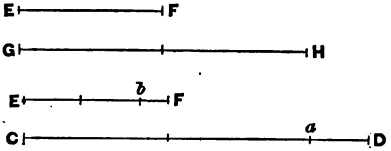

Fig. 4 *a*.

4. Applying EF twice to CD, I have a remainder *a*D; applying this twice to EF, I have a remainder *b*F, which is contained in AD twice. By adding the parts, we see that EF : CD :: 5 : 12.

Page 15, 39.

1. No.
2. No. Nearer B.
3. Yes. Fig. 20 shows how.

Page 17, 42.

1. Through all points which lie in the same direction from A as B lies. No.

2. Once.

3. Yes.

4. By driving out of the line joining the two places for a part of the distance, and then turning the horse around with his head away from Detroit. In either, or both. By driving on any curved line, which is always concave towards the line AD. A straight line.

Page 18, 44.

1. Toward C. An oblique line. Acute. Obtuse. Right.
2. Toward B. An oblique line. Obtuse. Acute. Right.

Page 19, 44.

1. After having determined B and C as before, draw two marks to intersect at a point D above XY, and join OD, producing it to meet XY at A.

2. Let him apply his pole at O, and notice where it touches the wall C D, first on one side, and then on the other, of the proposed cross-wall. Let him, by means of his pole, find the middle point of the line joining these points. This will be the point required.

ABOUT CIRCLES.

Page 20, 54.

1. A circle. A circumference. A circumference. An arc. The radius.

2. A circumference. The radius. A circle.

3. A circumference. The radius. The radius. An arc.

4. A circumference. No. The circle is the surface enclosed. A line.

5. In two. In two.

6. One cuts the circle, is a secant; the other just touches it, is a tangent.

7. A tangent.

8. The one 30 feet long.

9. Yes. No. One within the other. They would just touch externally. One would be wholly without the other.

10. A tangent.

11. No. More.

Page 22, 56.

1. Draw a circle with radius two inches, and then apply the radius as a chord six times, exactly as in Fig. 31.

2. Draw a circle with radius one inch; apply the radius as a chord six times, and join the alternate angles of the resulting hexagon.

Page 24, 58.

3. Let A, B, and C, be three points in the arc; then proceed exactly as in Fig. 33.

Page 24, 59.

1. $4 \times 3.1416 = 12.5664$.
2. $7 \times 3.1416 = 21.9912$. $21.9912 + 9 = 31$ in. nearly.
3. $\frac{288 \times 5280}{6 \times 3.1416} = 80672.27$ nearly.
4. $\frac{140 \times 33}{2 \times 200 \times 3.1416} = 3.676+$.
5. 1. $\frac{1}{4}$. $\frac{1}{8}$. The whole circumference.

ABOUT ANGLES.

Page 26, 61.

1. Let O be the acute angle, C the obtuse; *ab* and *cd* their measuring arcs. Applying *ab* to *cd*, I find it contained twice, with a remainder *fd;* applying this to *ab*, I find it contained four times. ∴ *ab* : *cd* :: 4 : 9; or, angle C is $2\frac{1}{4}$ times angle O.

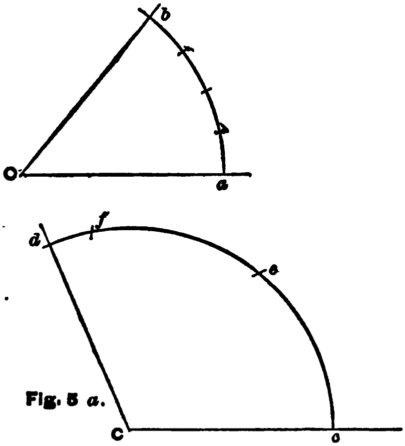

Fig. 5 *a.*

2. Let O and C be the angles, *ab* and *cd* their measuring arcs. Applying *ab* to *cd* three times, I find a remain-

der *ed;* applying this twice to *ab*, I find no remainder. ∴ *ab* : *cd* :: 2 : 7; or, angle **C** is $3\frac{1}{2}$ times angle **O**.

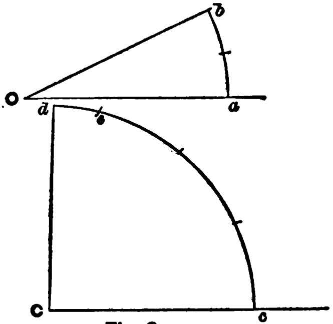

Fig. 6 *a*.

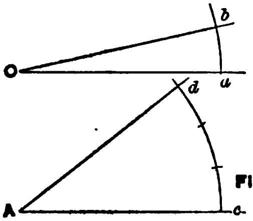

Fig. 7 *a*.

3. Let **O** be the given acute angle, *ab* any measuring arc. From **A** as a centre, with a radius **O***a*, strike an indefinite arc, and on it lay off *cd* equal to three times *ab*. Join *d* and **A**. Then will angle *c***A***d*, or **A**, be three times **O**.

4. Let **A** be the acute, **O** the right angle, *cd* and *ab* their measuring arcs. Applying *cd* to *ab*, I find a remainder *eb;* applying this to *cd*, I find a remainder *fd;* applying this to *eb*, I find a remainder *gb;* and applying this to *fd*, I find it contained twice. ∴ *cd* : *ab* :: 5 : 8; or, angle **O** is $1\frac{3}{5}$ times angle **A**.

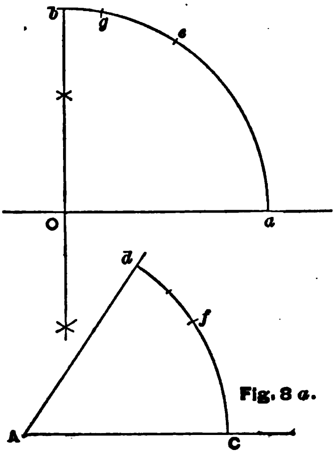

Fig. 8 *a*.

5. From the vertex of the angle as a centre, with any convenient radius, strike an arc intersecting the sides of the angle. Draw an indefinite straight line; at one extremity as a centre, with a radius equal to the one just used, strike an indefinite arc, and on this lay off from the line an arc

equal to the arc measuring the given angle. Joining the point thus determined with the centre, we have the angle required.

6. By constructing a right angle, it will be found that a is contained in it five times, in b twice, and in c three times. $\therefore$ $a = \frac{1}{5}$, $b = \frac{2}{5}$, and $c = \frac{3}{5} = .6$ of a right angle.

7. By constructing a right angle, it will be found that this right angle is contained in a once, with a remainder which is itself contained in the right angle three times. $\therefore$ $a = 1\frac{1}{3}$ times a right angle.

Applying a right angle to b, we shall find a remainder which is itself contained in the right angle twice. $\therefore$ $b = 1\frac{1}{2}$ times a right angle.

8. This example is exactly the same as the first part of the 7th.

9. Applying this angle to a right angle, we find a remainder which is itself contained in the given angle twice. $\therefore$ the given angle is $\frac{2}{3}$ of a right angle.

Page 27, 63.

1. All equal. Each measured by $\frac{1}{2}$ of AB.

2. All equal. Each measured by $\frac{1}{2}$ of a semi-circumference. $\frac{1}{4}$ of a circumference.

3. A semi-circumference. If c be an acute angle, we have more than a semi-circumference; if c be obtuse, we have less.

4. $\frac{1}{3}$. $1\frac{1}{3}$.

5. Each angle is measured by $\frac{1}{2}$ of $\frac{2}{6}$, or $\frac{1}{6}$ of a circumference.

Page 28, 64.

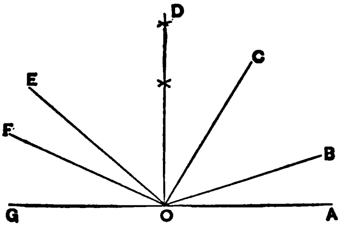

Fig. 9 *a*.

1. Acute. Obtuse. Right.

2. Let O be the point in GA, and AOB, BOC, COE, EOF, FOG, the angles at O. Draw OD perpendicular to AG at O. DOA and DOG are both right angles, and hence their sum is two right

angles. But their sum is the same as the sum of the angles **AOB**, **BOC**, **COE**, **EOF**, and **FOG**.

Page 29, 65.

1. Let **BAC** be the right angle. Lay off from **A**, equal distances, **A***b* and **A***c*. From *b* and *c* as centres, with any convenient radius, strike arcs intersecting at *d*. Then will **A***d* bisect the angle.

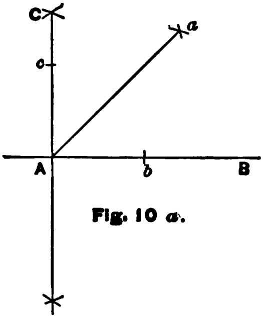

Fig. 10 *a*.

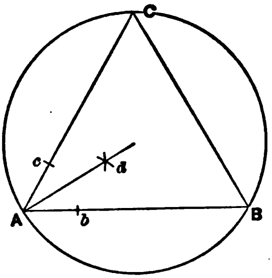

Fig. 11 *a*.

2. Inscribe the equilateral triangle in the circle by (55), and bisect the angle **A** as in solution of Example 1.

3. **EDB** is a right angle, being measured by $\frac{1}{2}$ of $\frac{3}{6}$ of a circumference. **FDB** is $\frac{2}{3}$ of a right angle. ∴ **EDF**, the difference, is $\frac{1}{3}$ of a right angle.

Page 30, 70.

1. By seeing how far it is straight across the farm at various points.

2. No.

3. 8. 4. Yes. When the line is perpendicular to the two parallels. When the line is oblique to them.

ABOUT TRIANGLES.

Page 31, 74.

1. One right angle.

2. No. Nothing, or zero.

3. No.

4. That the sum of the angles of a triangle is two right angles.

5. No. Otherwise the sums would not be equal to two right angles.

Page 33, 75.

1. The greatest. The least.

2. 3. 4. 5. 6. No.

7. They are unequal, and the greater is opposite the greater angle.

Page 34, 78.

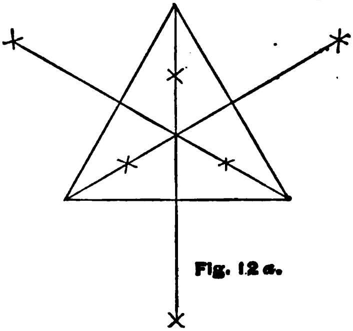

Fig. 12 a.

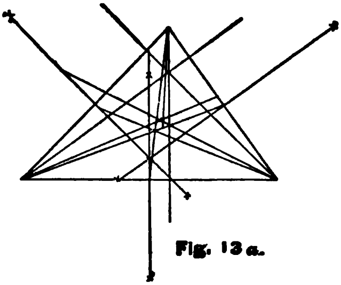

Fig. 13 a.

1. All in one place.

2. In different places.

ABOUT EQUAL FIGURES.

Page 35, 82.

1. No. Yes. Yes. No.

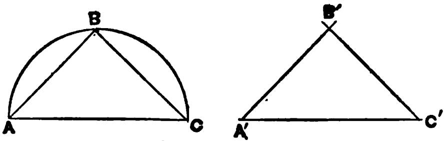

Fig. 14 a.

2. No. Yes. The triangles **ABC** and **A'B'C'** are right-angled, isosceles, and equal. Their construction is manifest in FIG. 14*a*.

Page 37, 85.

1. On **DF**. **BA**. No. On the outer side. No.

2. No. At **F**. **CB**. Without.

3. [The angles **C** and **E** are assumed equal.] Yes. Putting **D** on **A**, let **DE** take the direction **AC**. Now, as **DE** = **AC**, **E** will fall at **C**. As angle **E** = angle **C**, **EF** will take the direction **CB**; and, since **EF** = **CB**, **F** will fall at **B**. Hence the triangles coincide. No. Yes. Putting **F** on **A**, let **FE** take the direction **AC**. Since **FE** = **AC**, **E** will fall at **C**. Now, as angle **E** = angle **C**, **ED** will take the direction **CB**; and, since **ED** = **CB**, **D** will fall at **B**. No. The angle **E** being greater than angle **A**, if **EF** and **AC** coincide, **ED** will fall without **AB**.

5. Putting **C** on **H**, let **CD** take the direction **HG**. Since **CD** = **HG**, **D** will fall at **G**. As angle **D** = angle **G**, **DA** will take the direction **GF**; and, since **DA** = **GF**, **A** will fall at **F**. Now, as angle **A** = angle **F**, **AB** will take the direction **FE**; and, since **AB** = **FE**, **B** will fall at **E**.

6. Yes. Let **ABC** and **DEF** be the equilateral triangles. Putting **B** on **E**, let **BC** take the direction **ED**. Since **BC** = **ED**, **C** will fall at **D**. As angle **C** = angle **D**, **CA** will take the direction **DF**; and, since **CA** = **DF**, **A** will fall at **F**.

Page 39, 86.

2. Putting **A** on **H**, let **AC** take the direction **HE**. As **AC** = **HE**, **C** will fall at **E**. But, since angles **C** and **E** are unequal, **CD** and **EF** cannot coincide.

3. Yes. Putting **A** on **H**, let **AB** take the direction **HG**. Since **AB** = **HG**, **B** will fall at **G**. Now, as angle **B** = angle **G**, **BD** will take the direction **GF**; and, since **BD** = **GF**, **D** will fall at **F**. Again, as angle **A** = angle **H**, **AC** will take the direction **HE**. Since **AC** = **HE**, **C** will fall at **E**, and the trapeziums coincide.

ABOUT SIMILAR FIGURES, ESPECIALLY TRIANGLES.

Page 40, 88.

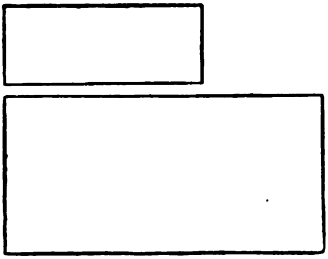

Fig. 15 *a*.

1. No. No.
2. Yes.
3. No. See Fig. 15*a*.

Page 41, 89.

1. Draw *cb*, and through *a* a parallel to it. This parallel will cut off on OY a distance equal to E.

2. Draw two indefinite lines OX and OY. Lay off O*c* equal to A, O*a* equal to B, O*f* equal to C. Join *c* and *f*, and through *a* draw a parallel to *cf*. O*d* will be the fourth proportional. Draw O′X′ and O′Y′. Lay off O′*c*′ equal to A, O′*b*′ equal to C, O′*d*′ equal to B. Join *b*′ and *d*′, and through *c*′ draw a parallel to *b*′*d*′. O′*e*′ will be the fourth proportional. In the upper figure lay

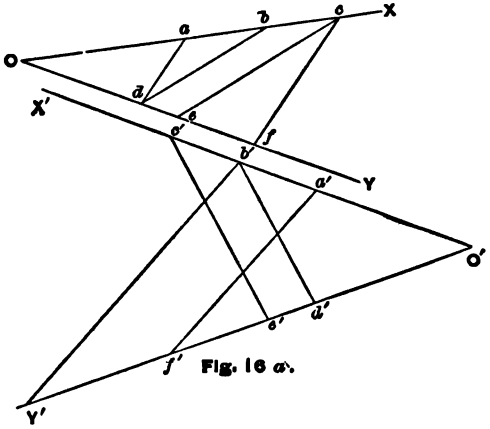

Fig. 16 *a*.

off O *b* equal to C, and O *e* equal to B. Join *ce*, and through *b* draw a parallel to *ce*. This will be found to intersect O Y at *d*. In the lower figure lay off O′*a*′ equal to B, O′*f*′ equal to A. Join *a*′*f*′, and through *b*′ draw a parallel to *a*′*f*′, which will be found to cut off on O′Y′ a distance nearly equal to 2 C.

Page 42, 90.

1. Lay off A′B′ equal to 8. At A′ and B′ respectively, construct angles equal to A and B, producing their sides to intersect at C′. Then will A′B′C′ be the triangle desired.

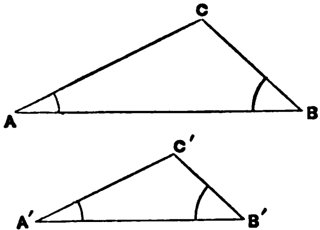

Fig. 17 *a*.

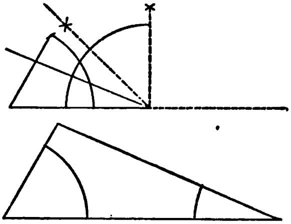

Fig. 18 *a*.

3. The construction of the figure is easily seen. 1 : 2. 1 : 2.

4. Yes. Those opposite 5 and 15, 8 and 24, 3 and 9. 5 and 15, 8 and 24, 3 and 9.

5. ADC and ACB. Homologous angles A and A, ADC and ACB, ACD and ABC. Homologous sides AC and AB, AD and AC, DC and CB. BDC and ACB. Homologous angles B and B, BDC and ACB, BCD and BAC. Homologous sides BC and AB, CD and AC, DB and CB. ADC and BDC. Homologous angles ADC and BDC, ACD and DBC, CAD and DCB. Homologous sides AD and DC, DC and DB, AC and CB. Yes.

6. AB : AC :: BD : CE, or 10 : 80 :: 5 : 40.

7. No. Yes.

8. Yes. Yes.

9. No. Yes. Yes.

10. Yes. No. No.

11. Because triangles are thus made, whose form is unchangeable. No. No.

ABOUT AREAS.

Page 46, 100.

2. Lay off **A B** equal to 2 inches; at **A** construct an angle equal to $\frac{2}{3}$ of a right angle. Make **A D** = 2 inches. From **D** and **B** as centres, with a radius 2 inches, strike arcs intersecting at **C**. **A B C D** will be the rhombus required.

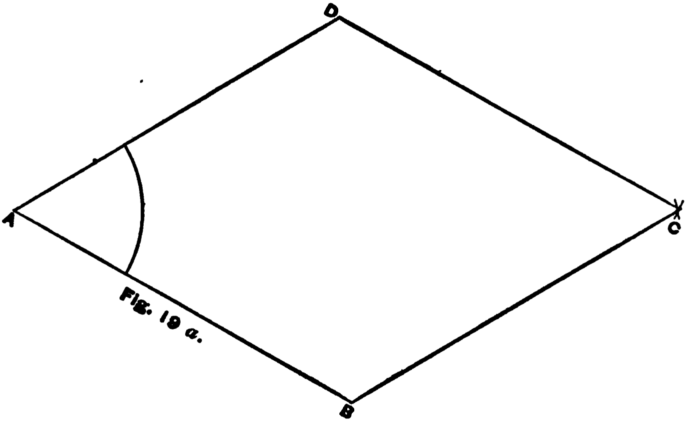

Fig. 19 a.

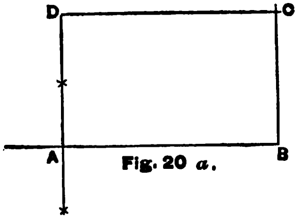

Fig. 20 a.

3. Draw an indefinite line **A B**; at **A** draw a perpendicular, and lay off **A D** = 3. Make **A B** = 5. From **B** as a centre, with radius 3, and from **D** as a centre, with radius 5, strike arcs intersecting at **C**.

4. Construct a right angle; bisect it; on one side of this angle lay off 3, and on the other lay off 7. From the ex-

tremity of side 3, with radius 7, and from the extremity of side 7 with radius 3, strike arcs intersecting so as to determine the 4th vertex.

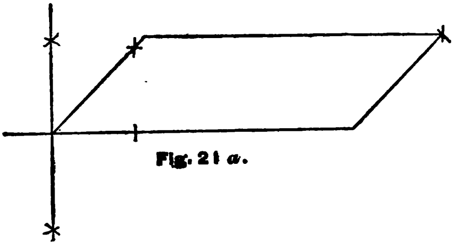

Fig. 21 a.

5. None. 2. 5. 9. 20. A figure of n sides, $\frac{1}{2}n(n-3)$.

Page 47, 104.

Ex. The longest. The largest.

Page 49, 108.

1. $20 \times 30 + 2 \times 20 \times 15 + 2 \times 30 \times 15 = 2{,}100$ sq. ft. $= 233\frac{1}{3}$ sq. yds.

2. The purpose of this exercise is to draw the pupils into a discussion which shall fix in their minds the fundamental principle in the doctrine of areas, viz.: That parallelograms of equal bases and equal altitudes are equivalent. The correctness of the answer in the text will be seen if we consider that the *intrinsic* value of the velvet is $8 per yard. Hence the salesman is right "in his *demand*," since the original form of the piece was a rectangle, and there has been a waste of a square ($\frac{1}{2}$ yd.) in cutting it "on the bias." On the other hand, the customer is right "in his *statement*," which is, that he is getting only $\frac{1}{2}$ a yard of velvet.

3. Neither.

4. All have the same area.

5. The one with the vertex directly above the centre of the semicircle.

6. No. Yes.

7. Rectangle.

8. $\frac{12 \times 16 \times 10}{12} = 160.$ $\frac{15 \times 11 \times 9}{12} = 123\frac{3}{4}.$

$\frac{8 \times 10 \times 13}{12} = 86\frac{2}{3}.$ **(108.)**

9. $\frac{6 \times 10}{2} = 30.$ No.

10. They are equal, since the triangles have equal bases and the same altitude.

PYTHAGOREAN PROPOSITION.

Page 50, 109.

2. Yes. $9^2 + 12^2 = 81 + 144 = 225 = 15^2.$

3. 8. $\sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8.$

4. $\sqrt{10^2 - 4^2} = \sqrt{100 - 16} = \sqrt{84} = 9.165 +.$

5. $\sqrt{20^2 + 30^2 + 15^2} = \sqrt{400 + 900 + 225} = \sqrt{1525} = 39.05 +.$

6. Yes.

7. In the acute-angled triangle $C''B = CB$, but $\overline{AC''}^2$ is less than $\overline{AC}^2$. $\therefore\ \overline{AC''}^2 < \overline{BC''}^2 + \overline{AB}^2.$

Page 52, 110.

4. Draw an indefinite line AX; on it lay off $AB = 3$. At A erect a perpendicular, and on it lay off $AD = 2$. From B and D as centres, with radii 2 and 3 respectively, strike arcs intersecting at C. Thus we have the rectangle. On an indefinite line lay off $EF = 3$, and $FG = 2$; on EG as a diameter, draw a semi-circumference. At F erect the perpendicular FH, and construct the square having this line as a side. It will be equivalent to the rectangle.

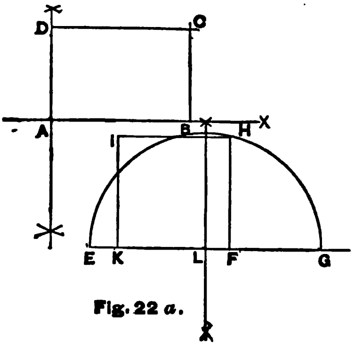

Fig. 22 a.

Page 52, 111.

1. A's land : B's land :: $5^2 : 3^2 :: 25 : 9$. $\therefore$ A's $= 2\frac{7}{9}$ B's.
2. As 2 to 1.
3. As 2 to 3.

THE AREA OF A CIRCLE.

Page 54, 113.

1. If 1 is the side of the square, the area of the circle will be $(\frac{1}{2})^2 \times 3.1416 = .7854 = 1 - .2146$.

2. $\frac{3^2 \times 3.1416}{4} = 7.0686$. $7.0686 \times 640 = 4523.904$.

3. $100^2 \times 3.1416 = 31{,}416$ sq. ft. In $\frac{3}{4}$ of an acre there are 32,670 sq. ft.

4. 3.1416.

5. $4 \times 3.1416 = 12.5664$. $9 \times 3.1416 = 28.2744$. $16 \times 3.1416 = 50.2656$. Four times, nine times, sixteen times as great.

Page 55, 114.

1. The area of the outer is four times that of the inner. The four parts are equivalent.

2. $3^2 : 5^2 :: 9 : 25$. $\frac{25}{9} = 2\frac{7}{9}$.

3. $2 \times 5^2 = 2 \times 25 = 50$. $\sqrt{50} = 7.071$.

4. Yes. Four times as great.

5. Not quite. They are as 49 to 50.

6. None.

Page 57, 121.

1. 3. 6. 10. 7. 4. See Figs. 23 *a*–27 *a* inclusive. In general $n - 2$, n being the number of sides.

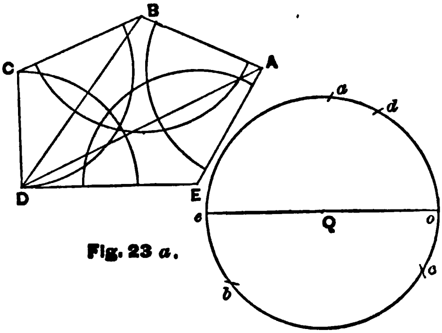

Fig. 23 *a*.

2. 6 × 2 − 4 = 8. Let **ABCDEF**, Fig. 27*a*, be a hexagon. With any radius, as **QO**, describe a circumference. With the same radius describe arcs measuring the angles **A, B, C, D, E,** and **F.** Beginning at **O**, lay off **O***a*, the measure of **A**, *ab* of **B**, *bc* of **C**, *cd* of **D**, *de* of **E**, and *ef* of **F**, when *f* will be found to coincide with **O**. The sum of the angles, being measured by two circumferences, is eight right angles. 5 × 2 − 4 = 6. Let **ABCDE**, Fig. 23*a*, be any pentagon. With any radius as **QO**, describe a circumference, and also arcs measuring the angles **A, B, C, D,** and **E.** Beginning at **O**, lay off **O***a*, the measure of **A**, *ab* of **B**, *bc* of **C**, *cd* of **D**, *de* of **E**, when *e* will be found to fall at the extremity of the

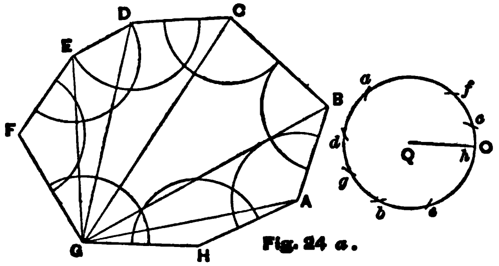

Fig. 24 *a*.

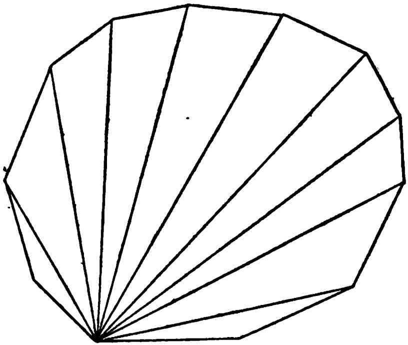
Fig. 25 *a*.

diameter through O. The sum of the angles is thus seen to be six right angles. $8 \times 2 - 4 = 12$. Let ABCDEFGH, FIG. 24*a*, be any octagon. With any radius, as QO, describe a circumference, and also arcs measuring the angles as before. Beginning at O, lay off O*a*, the measure of A, *ab* of B, *bc* of C, *cd* of D, *de* of E, *ef* of F, *fg* of G, and *gh* of H, when *h* will fall at O. The sum of the angles is thus seen to be twelve right angles. $10 \times 2 - 4 = 16$. Let ABCDEFGHIK, FIG. 28*a*, be any decagon. With any radius, OQ, describe a

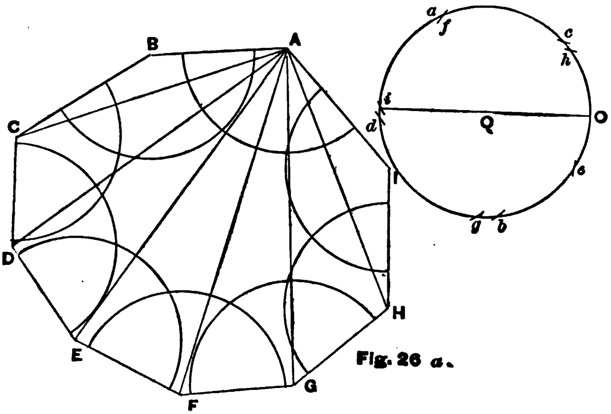

Fig. 26 *a*.

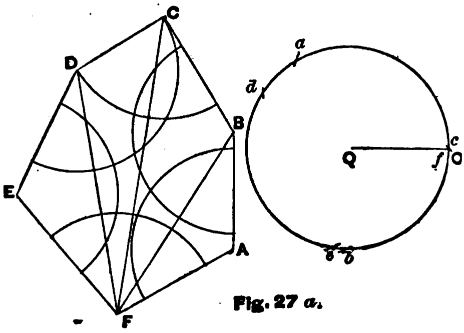

Fig. 27 *a*.

circumference, and also arcs measuring the angles. Beginning at **O**, lay off the arcs **O**a, ab, bc, cd, de, ef, fg, gh, hi, ik, when k will fall at **O**, thus showing that the sum of the angles is sixteen right angles. $9 \times 2 - 4 = 14$. Let **ABCDEFGHI**, Fig. 26a, be any nonagon. With any radius, as **QO**, describe a circumference, and also arcs measuring the angles. Beginning at **O**, lay off **O**a, ab, bc, cd, de, ef, fg, gh, hi, when i will fall at the end of the diameter through **O**, thus showing that the sum of the angles is fourteen right angles. The sum of the angles of a triangle has been shown to be two right angles in Fig. 53. That the sum of the angles of a quadrilateral is four right angles, may be seen from Fig. 29a.

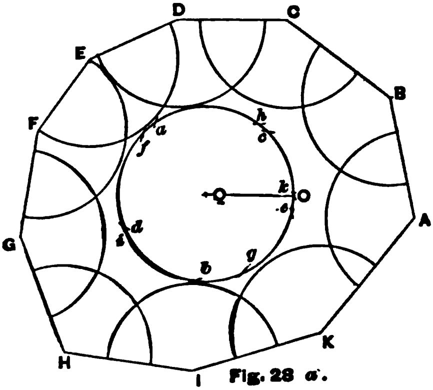

Fig. 28 a.

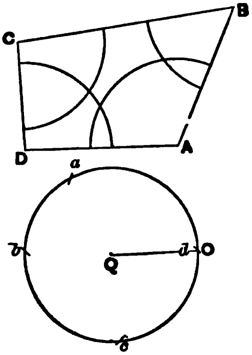

Fig. 29 a.

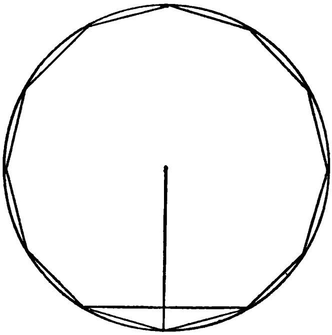

Fig. 30 a.

3. Since the sum of the angles is eight right angles, each is equal to $\frac{8}{6}$ of a right angle, or $1\frac{1}{3}$ right angles.

4. In the octagon each angle equals $\frac{12}{8} = \frac{3}{2}$ of a right angle, or $1\frac{1}{2}$ right angles. Similarly

each angle of a pentagon equals $1\frac{1}{5}$ right angles, and each angle of a dodecagon equals $1\frac{2}{3}$ right angles.

5. Describe a circle with any radius. Obtain one side of a regular inscribed hexagon, bisect the subtended arc, and lay the half off twelve times, as is shown in Fig. 30*a*.

7. Draw a circle with any radius. Draw two diameters at right angles, bisect the intercepted arcs, and join the points of bisection.

10. As in Fig. 31*a*, draw a circumference, and two diameters at right angles. At the extremities of each diameter, draw lines parallel to the other, and we shall have the square desired.

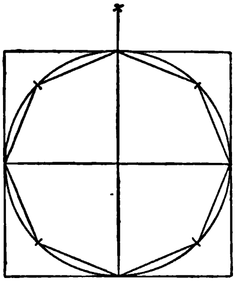

Fig. 31*a*.

To circumscribe an equilateral triangle. Having drawn a circle, find the vertices of a regular inscribed hexagon. At the 1st, 3d, and 5th, draw lines touching the circumference, until they meet. To circumscribe a regular hexagon, proceed as before to determine the vertices of the regular inscribed hexagon. At these vertices draw lines touching the circumference, until they meet.

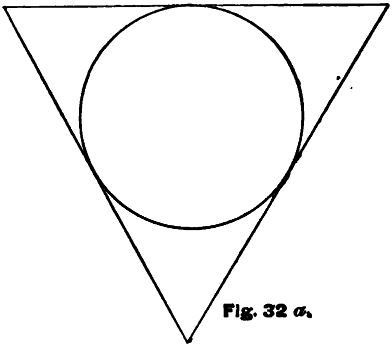

Fig. 32 *a*.

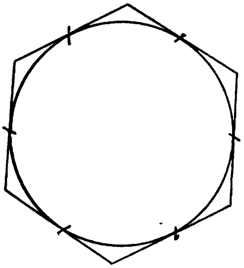

Fig. 33 *a*.

PART II.

PLANE GEOMETRY.

OF PERPENDICULAR STRAIGHT LINES.

Page 64, 130.

4. P and P′ are each equally distant from A and B.

5. The latter. (128.)

6. At different points. (127.) Perpendicular, or at right angles, to the road.

7. Lose. Referring to Fig. 107, in the two paths AOB and AO′B, AO is common, but $OO' + O'B > OB$. Gain. In same figure, suppose O″ to be a point between A and O. Then $AO'' + O''B < AO'' + O''O + OB$.

OF OBLIQUE STRAIGHT LINES.

Page 69, 141.

1. Having an angle given, produce either of the two sides backward through the vertex.

2. In one and the same straight line. Because the sum of the angles is two right angles. See Fig. 34 *a*.

3. Because the sum of the angles is four right angles.

4. In one and the same straight line. (136.)

5. (137) or (141).

6. A. (139.)

OF PARALLELS.

Page 75, 156.

3. Proceed as in Fig. 163, noticing that the measuring arcs may be drawn with a radius different from P*a*.

4. Parallel. (**149.**) Move the tongue until the same line shall be given, whether the edge *m* or *m′* be applied. $\frac{1}{2}$ a right angle.

5. No. Because the plumb lines, if produced, would meet at the centre of the earth.

6. Referring to FIG. 33, since MN is perpendicular to AB at its middle point, every point in it is equally distant from A and B (129). Again, for the same reason, every point in RS is equally distant from B and C. Consequently their intersection O is a point equally distant from the three points A, B, and C.

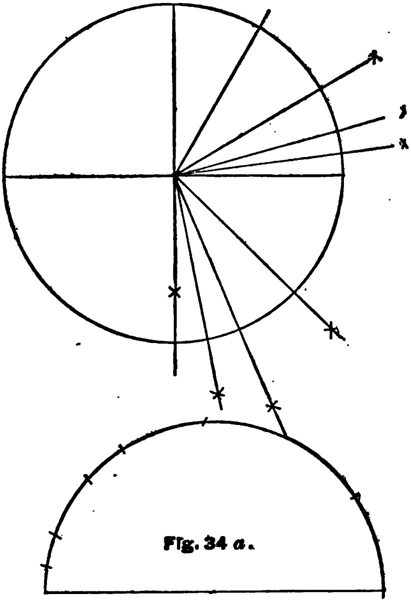

Fig. 34 *a*.

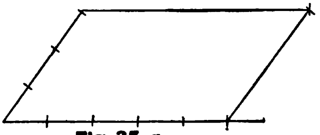

Fig. 35 *a*.

7 and 8. See FIG. 234. They are shaped alike.

9. Yes.

OF THE RELATIVE POSITIONS OF STRAIGHT LINES AND CIRCUMFERENCES.

Page 84, 176.

1. Draw a circle, and any diameter, as in FIG. 131. **(158.)**

2. Draw a diameter through the point. **(158.)**

4. **(160.)**

5. The one at 4. **(163.)**

6. The same. **(164.)**

7. A secant. A tangent. Wholly outside.

8. From any point, as O, with a radius O P, strike an arc passing through the given point P, and intersecting the given line A B at *a* and *b*. From *b* as a centre, with a radius equal to *a* P, strike an arc cutting the circumference at *c*. Then by **(174)** and **(165)**, P *c* is parallel to A B.

10. **(159.)**

OF THE RELATIVE POSITIONS OF CIRCUMFERENCES.

Page 92, 199.

1. Reasons given in **(177)**.

2. Join the points by a straight line. Bisect this line by a perpendicular. Where this perpendicular meets the given line will be the required centre, since it is in the given line and equally distant from the two points **(129)**.

3. Reasons given in **(177)**.

4. They intersect. They are tangent externally. One wholly exterior to the other. One wholly within the other. Concentric. One tangent to the other internally.

5. Join O and O′, bisect that portion of O O′ intercepted by the circumferences, and with the middle point of this portion as a centre, and half of it as a radius, describe a circumference. This will be tangent to the other two externally, since in each case the distance between the centres is equal

to the sum of the radii. Again, produce **O O′** in both directions till it intersects the circumferences. On this line as a diameter, describe a circumference to which the other two will be tangent internally, since in each case the distance between the centres is equal to the difference of the radii.

In another way, take some definite line as a radius for the required circle; from **O** as a centre, with a radius equal to the sum of the desired radius and the radius of **O**, strike an arc; from **O′** as a centre, with a radius equal to the sum of the desired radius and the radius of **O′**, strike an arc intersecting the former arc in two points. These points will be the centres of circles tangent to the two given circles externally. By using, in a similar way, as radii, the differences between the desired radius and the radii of **O** and **O′** respectively, we may obtain a circumference to which **O** and **O′** shall be tangent internally.

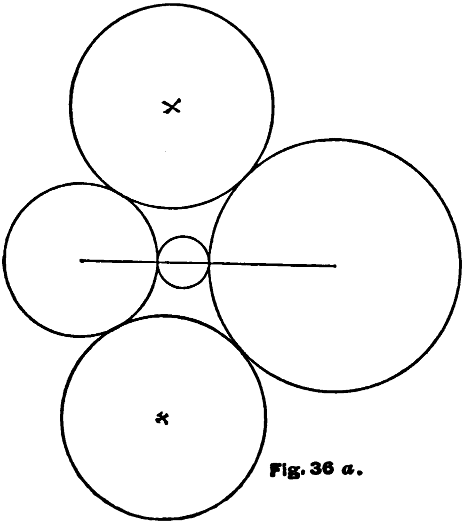

Fig. 36 *a*.

6. For method see (5). Yes. Its centre will lie in the line joining the centres of circles 6 and 10. No. Because the arcs struck from the centres of circles 6 and 10, with radii 6 + 1, and 10 + 1, will not intersect.

OF THE MEASUREMENT OF ANGLES.

Page 102, 218.

2. **CDB** and **CAB**, **ADC** and **ABC**, **ACD** and **ABD**, **DAB** and **DCB**. Equal because they have equal measures. **ABD** is measured by $\frac{1}{2}$ of arc **AD**, **CDB** by $\frac{1}{2}$ of arc **CB**, while **COB** is measured by $\frac{1}{2}$ (arc **CB** + arc **AD**). In a similar way, the measure of **DOB** is the sum of the measures of **ABC** and **DAB**.

4. Second Solution. Angle **ADB** = angle **ECB** = angle **O** (152). Any angle inscribed in the segment **A*m*B** = angle **O** (211). By making **CBA** *acute*, we can always render **CBA** + **ECB** < 2 right angles, and hence **ADB** will be a possible triangle. When angle **O** is given, the angle **ABD** may be made any thing less than the supplement of **O**. Hence, in some cases, **ABD** may be right, or even obtuse.

OF THE ANGLES OF POLYGONS, AND THE RELATION BETWEEN THE ANGLES AND SIDES.

Page 118, 276.

2. $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$. $2 - \frac{7}{6} = \frac{5}{6}$.

3. The side opposite $\frac{5}{6}$ of a right angle. Side opposite $\frac{1}{2}$ of a right angle. No.

4. $2 - \frac{1}{3} = \frac{5}{3}$. $\frac{1}{2}$ of $\frac{5}{3} = \frac{5}{6}$.

5. A trapezoid, since the remaining angles are each a right angle, and hence two of the sides are parallel.

6. One, $\frac{5}{3}$ of a right angle; the remaining two, each $\frac{1}{3}$.

7. Yes. Construct an angle equal to $\frac{2}{3}$ of a right angle, opposite it construct an angle equal to $\frac{4}{3}$ of a right angle, and produce the sides of the two angles until they intersect. Through three of the points or vertices of this figure, pass a circumference, and it will be found to pass through the fourth.

8. A parallelogram. No.

9. That it is a parallelogram.

10. $12 \times 2 - 4 = 20$. $\frac{20}{12} = 1\frac{2}{3}$.

11. $9 \times 2 - 4 = 14$. $\frac{14}{9} = 1\frac{5}{9}$. $2 - 1\frac{5}{9} = \frac{4}{9}$.

12. A figure of 17 sides, since $\frac{17 \times 2 - 4}{17} = 1\frac{13}{17}$.

13. A hexagon, since $\frac{2}{3} = \frac{4}{6}$.

14. Yes. Yes. Squares or rectangles. No. Since each angle of a regular pentagon is $1\frac{1}{5}$ right angles, we cannot group regular pentagons about a point so that the sum of the angles at the point shall be four right angles. Yes. Each angle of a regular hexagon being $1\frac{1}{3}$ right angles, three regular hexagons can be grouped about a point so as to cover the space exactly. The bee. No. Because in no such case will the sum of any number of angles of a regular polygon be exactly four right angles.

15. No. Yes. No. Yes.

16. Yes, to all.

17. Yes. Yes. Pentagon, yes; octagon, no.

OF EQUALITY.

Page 134, 310.

2. Draw an indefinite line. On this lay off the given side; at one end of this line, construct an angle equal to one of the given angles, at the other end an angle equal to the other given angle, and produce the sides until they meet.

6. Triangles **ACD** and **DCB** are mutually equiangular, since there is a right angle in each, and **CAD** = **DCB**, each being the complement of **ACD**. **ACD** and **ACB** are mutually equiangular, since there is a right angle in each, and angle **CAD** is common. Similarly with respect to **ACB** and **DCB**.

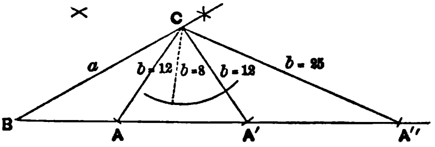

Fig. 37 *a*.

7. There is no triangle, since $8 + 5 < 15$.

8 and 9. Proceeding as in (5), we see that for $b = 8$,

there is no triangle; for $b = 12$, there are two; for $b = 25$, there is one.

10. Lay off **CB** $= a = 15$; at **B** construct the angle **B** $= \frac{1}{2}$ of a right angle; at **C** construct the angle **C** equal to the supplement of **A** + **B**, and produce **CA** and **BA** till they intersect. Angle **A** will then equal $\frac{2}{3}$ of a right angle.

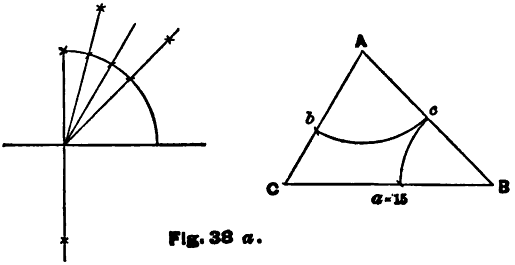

Fig. 38 *a*.

11. Lay off **AC** $= 16$, and on **AC** as a diameter, construct a semicircle. From **A** as a centre, with a chord equal to 7, strike an arc intersecting the circumference at **B**. Draw **AB** and **BC**. Again, draw the indefinite line **AC**; at **A** construct the given angle, lay off **AB** $= 12$, and at **B** draw a perpendicular, producing it to meet **AC**. This will give us one triangle. To construct the other, lay off **BD** $= 12$, and draw **DC′** parallel to **BA**; through **C′** draw **C′B′** parallel to **CB**. The two triangles will be alike in shape but different in size.

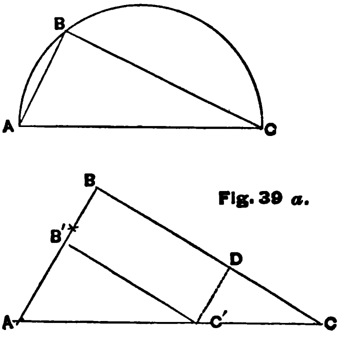

Fig. 39 *a*.

12. Lay off **AB** $= 20$; on **AB** as a diameter construct a semicircle. At **B** make an angle equal to $\frac{1}{3}$ of a right angle. Join **AC**. See Fig. 40 *a*.

14. Lay off **AB** = 8, at **A** construct an angle equal to $1\frac{1}{3}$ right angles, and lay off **AD** = 6. Now, through **D** draw **DC** parallel to **AB**, and, through **B**, **BC** parallel to **AD**, producing them to meet at **C**.

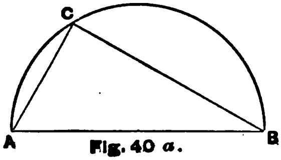

Fig. 40 *a*.

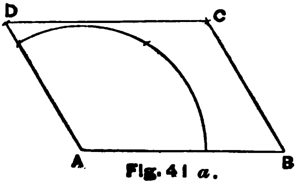

Fig. 41 *a*.

15. Proceeding as suggested in the book, we finally reach **F**. Now, if from **F** as a centre, with a radius equal to 3, and from **A** as a centre, with a radius equal to 4, we strike arcs, they, obviously, will not intersect. If, however, the sum of **AG** and **FG** be greater than **AF**, as when **FG**$_1$ = 7 and **AG**$_1$ = 6, the arcs will intersect as at **G**$_1$ and the heptagon be formed. The sum of the given angles cannot exceed eight right angles, and the sum of the sides included between the unknown angles must be greater than **AF**.

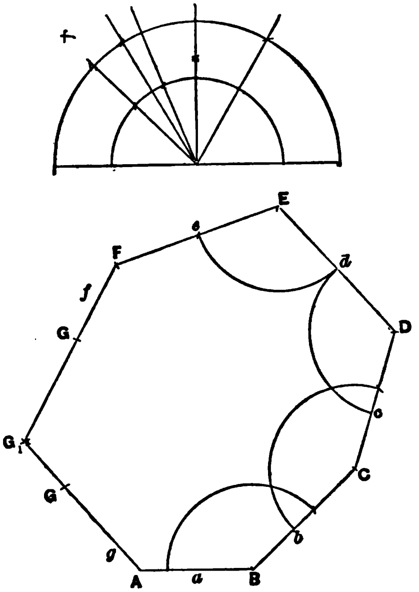

Fig. 42 *a*.

OF AREA.

Page 143, 331.

1. $\frac{75 \times 110}{2 \times 160} = 25.78+$ acres.

2. $\frac{126 \times 72}{2} = 4{,}536$ sq. ft.

3. Into parts which are to each other as 2, 3, and 5. Into two parts of equal area.

OF SIMILARITY.

Page 151, 353.

2. **(338.)**

4. Let AB be the given line. Draw AX making any convenient angle with AB; on it lay off five units of any length; join X, the fifth point of division, with B, and draw a series of parallels to BX through the various points of division. AB will thus be divided into five equal parts.

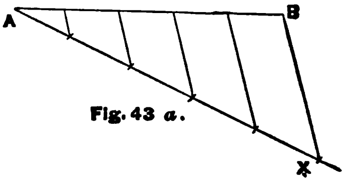

Fig. 43 *a*.

6. **(347.)**

7. ODE = OCB, OED = OBC **(152)**; DOE = BOC **(134)**. OD : OC :: OE : OB, OD : OE :: OC : OB, OD : DE :: OC : BC, OB : BC :: OE : DE, all true.

8. $b = \mathrm{B}$, $c = \mathrm{C}$, $d = \mathrm{D}$, $e = \mathrm{E}$, $f = \mathrm{F}$ **(152)**; $\mathrm{A}b : bc : cd : de : ef : f\mathrm{A}$ :: AB : BC : CD : DE : EF : FA **(338)**. Hence the angles of A*bcdef* are equal, and also the sides. For latter part, see **(363)**.

9. $9 : 11 : 7 :: 4\frac{1}{2} : 5\frac{1}{2} : 3\frac{1}{2}$.

10. $\sqrt{5 \times 15} = \sqrt{75} = 8.66+$. $\sqrt{20 \times 15} = \sqrt{(15)^2 + (8.66+)^2} = \sqrt{300} = 17.32+$. $\sqrt{20 \times 5} = \sqrt{5^2 + (8.66)^2} = \sqrt{100} = 10$.

APPLICATIONS OF THE DOCTRINE OF SIMILARITY.

Page 161, 379.

1. Through the point draw a diameter. The rectangle of the segments of each chord will be equal to the rectangle of the segments of the diameter, and hence be constant.

2. $\frac{4 \times 6}{3} = 8.$

3. The propositions may be stated in one, as follows: If two lines intersect each other and the circumference of a circle, the product of the distances from the intersection of the lines to the intersections of one with the circumference will be equal to the product of the corresponding distances upon the other. In Fig. 245, when **AO** becomes a tangent, $OD = AO$.

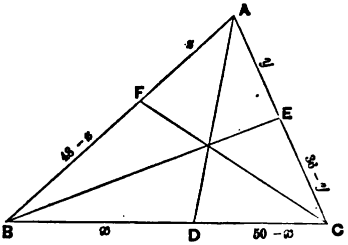

Fig. 44 *a*.

4 and 5. Let **ABC** be the triangle, $AB = 48$, $AC = 36$, $BC = 50$; then $48 : 36 :: 4 : 3 :: x : 50 - x$, or $4 : 7 :: x : 50$. $\therefore\ x = \frac{200}{7} = 28\frac{4}{7}$. $50 - x = 21\frac{3}{7}$. Again, $48 : 50 :: 24 : 25 :: y : 36 - y$, or $24 : 49 :: y : 36$. $\therefore\ y = \frac{24 \times 36}{49} = 17\frac{31}{49}$. $36 - y = 18\frac{18}{49}$. Again, $36 : 50 :: 18 : 25 :: z : 48 - z$, or $18 : 43 :: z : 48$. $\therefore\ z = \frac{18 \times 48}{43} = 20\frac{4}{43}$. $48 - z = 27\frac{39}{43}$. $AB \times BC = AE \times EC + \overline{BE}^2$. $AB \times AC = BD \times DC + \overline{AD}^2$. $BC \times AC = BF \times AF + \overline{CF}^2$. $\therefore\ BE = \sqrt{AB \times BC - AE \times EC} = \sqrt{48 \times 50 - 17\frac{31}{49} \times 18\frac{18}{49}} = 45.56$. $AD = \sqrt{AB \times AC - BD \times DC} =$

$$\sqrt{48 \times 36 - 28\tfrac{4}{7} \times 21\tfrac{3}{7}} = 33.41+.$$

$CF = \sqrt{BC \times AC - BF \times AF} =$

$$\sqrt{36 \times 50 - 27\tfrac{39}{43} \times 20\tfrac{4}{43}} = 35.20+.$$

7. $30^\circ = \dfrac{2 \times 20 \times 3.1416}{12} = 10.472.$

$$\text{Area} = \frac{(20)^2 \times 3.1416}{12} = 104.72.$$

8. In order that a given circle may be divided into parts equal in area by concentric circumferences, it is evident that the areas of the circles must form an arithmetical progression whose first term and common difference are the area of the inner circle, i.e., the areas of the successive circles will be $1, 2, 3, 4 \ldots\ldots n$: hence their radii will be as $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots\ldots \sqrt{n}$. If the radius of the outer be r, of the inner r', we have $r : r' :: \sqrt{n} : 1$. $\therefore r' = \dfrac{r}{\sqrt{n}}$, or, $2r' = \dfrac{2r}{\sqrt{n}}$. The successive diameters will be $\dfrac{2r}{\sqrt{n}}, \dfrac{2r\sqrt{2}}{\sqrt{n}}, \dfrac{2r\sqrt{3}}{\sqrt{n}}, \ldots\ldots 2r$.

When $r = 24$ and $n = 5$, the successive diameters will be $\frac{48}{5}\sqrt{5}, \frac{48}{5}\sqrt{10}, \frac{48}{5}\sqrt{15}, \frac{96}{5}\sqrt{5}, 48$.

11. If through the point nearer to the line of projection, we draw a parallel to that line, a right-angled triangle is formed of which the first line is the hypotenuse, its projection one side, and the difference of the perpendiculars the other. Hence the demonstration.

12. If from any vertex of a triangle a perpendicular be drawn to the opposite side, the sum of the segments thus formed is to the sum of the other two sides as the difference of those sides is to the difference of the segments. If the perpendicular falls without, instead of "segments," read "distances from the foot of the perpendicular to the extremities of that side."

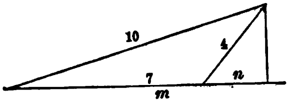

Fig. 45 a.

13. $6 : 5 + 4 :: 5 - 4 : m - n$, or, $6 : 9 :: 2 : 3 :: 1 : m - n$. $\therefore m - n = 1.5$. Now, $m + n = 6$. $\therefore m = 3.75$; $n = 2.25$.

14. $7 : 10 + 4 :: 10 - 4 : m - n$, or $1 : 2 :: 6 : m - n$. $\therefore\ m - n = 12$. $m + n = 7$. $\therefore\ m = 9.5$, $n = -2.5$. n is negative because the perpendicular falls without.

15. Referring to FIG. 95, it is evident that the side of the inscribed square is $\sqrt{2}$. Now, from the formula $c = \sqrt{2 - \sqrt{4 - C^2}}$, one side of the octagon is $\sqrt{2 - \sqrt{2}}$. The perimeter is $8\sqrt{2 - \sqrt{2}} = 6.123$. The apothem is

$$\sqrt{1 - \left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^2} = \tfrac{1}{2}\sqrt{2 + \sqrt{2}};$$

$\therefore$ The area $= \tfrac{1}{4}\sqrt{2 - \sqrt{2}} \times \sqrt{2 + \sqrt{2}} \times 8 = 2\sqrt{2} = 2.8284+$. When the radius is 10, the perimeter $= 61.23$, and the area $= 282.84+$.

16. From $c = \sqrt{2 - \sqrt{4 - C^2}}$. $c^4 - 4c^2 + 4 = 4 - C^2$, or $C^2 = -c^4 + 4c^2$. $\therefore\ C^2 = -1 + 4 = 3$, and $C = \sqrt{3}$.

SOLID GEOMETRY.

OF STRAIGHT LINES AND PLANES.

Page 173, 419.

1. Imagine a line through the corner of the desk and the point on the stove; then conceive any plane containing this line to revolve about it until the point in the ceiling falls in the plane.

2. Two lines which lie in parallel planes, but through which a single plane cannot be passed.

3. See the preceding. A line drawn between the lines so as to be perpendicular to both.

4. From the point in the ceiling as a centre, with the 12-foot pole as a radius, describe a circumference on the floor. The centre of this circumference will be the point desired.

5. A plane can always be passed through three points not in the same line, and is completely determined thereby. It may or may not pass through four.

6 and 7. Place one square against a side of the stud and the floor, the second against the adjacent side of the stud and the floor, and then move the stud till it comes directly against both squares, thus forming a right angle with two lines of the floor. Two opposite sides of the stud would not answer.

8. Pass a plane through the eye and the given line; its intersection with the plane of projection is evidently the projection desired, and a straight line by (**386**). Draw two perpendiculars measuring the distances between the parallels at any two points. Pass planes through the eye and the sides of the rectangle thus formed so as to intersect the plane of projection. If now the parallels are parallel to the plane of projection, we can by similar triangles show that the figure formed is still a rectangle; if, however, the two parallels are not parallel to the plane of projection, we can by the same means show that the figure formed will no longer have the two projections parallel.

9. The angle may be any thing between 23° and its supplement, or 157°.

OF SOLID ANGLES.

Page 185, 456.

1. A right angle. At any acute angle. At any obtuse angle.

2. Because the eye is not in a plane perpendicular to each one of the edges, and hence the apparent angle is greater or less than a right angle. No.

3. In a plane perpendicular to the edge. No, in general.

4. A supplementary triedral. The one is the supplement of the other. The opposite ones.

5. Yes. In two: all three may intersect in the same straight line, or every two may intersect in parallel lines. If they have only one common point, a triedral is formed.

6. The triedrals are symmetrical, not equal.

7. No. Because the sum of two of the facial angles is less than the third. Yes, provided the sum of every two be greater than the third. The triedral then becomes a plane.

OF PRISMS AND CYLINDERS.

Page 198, 499.

1. As 6 to 8, or 3 to 4. (487.)

2. $2 \times 6 \times 4 + 2 \times 4 \times 5 + 6 \times 5 = 118$ sq. ft.

3. No. Yes. Eight times as much.

4. $(7 \times 3.1416 + 1.5) \times 28 = 657.75+$.

5. $3.1416 \times \frac{1}{64} \times 24 \times 60 \times 3 = 212.058$ cu. ft. Four times as much.

6. $\sqrt[3]{125} : \sqrt[3]{8} :: 5 : 2$.

7. $5^2 : (5\frac{3}{4})^2 : 6^2 :: 25 : 33\frac{1}{16} : 36$. $5^3 : (5\frac{3}{4})^3 : 6^3 :: 102.51+ : 156 : 177.24+$.

8. $(5.5)^3 : (5.75)^3 :: 160 : 182.82+$. The former is the stouter.

9. Both have the same volume and the same lateral surface.

10. $12 \times 3.1416 \times \frac{1}{4} \times (\frac{29}{12})^2 = 55.0434$ cu. ft. $55.0434 \times \frac{3}{4} \times 12 = 495.39$ sq. ft.

OF PYRAMIDS AND CONES.

Page 208, 531.

1. Each side of the hexagonal base $= 10$. Apothem $= \sqrt{(10)^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$. Area base $= \frac{1}{2} \times 60 \times 5\sqrt{3} = 150\sqrt{3}$. Volume $= \frac{1}{3} \times 8 \times 150\sqrt{3} = 400\sqrt{3} = 692.82$. Slant height $= \sqrt{8^2 + 75} = \sqrt{139}$. Lateral surface $= \frac{1}{2} \times 60 \times \sqrt{139} = 353.69$.

2. In a circle whose radius is 1, if x be the side of a regular inscribed decagon, $1 : x :: x : 1 - x$ (Ex. 10, p. 162), whence $x = \frac{1}{2}(\sqrt{5} - 1)$. Now by (372) $c = \sqrt{2 - \sqrt{4 - c^2}}$,

whence $\mathrm{C} = c\sqrt{4 - c^2}$. Substituting for c, $\frac{1}{2}(\sqrt{5} - 1)$, we have C, the side of a regular pentagon inscribed in a circle whose radius is 1, equal to $\frac{1}{2}\sqrt{10 - 2\sqrt{5}}$. $\therefore$ Each side of the pentagonal base under consideration is equal to $3\sqrt{10-2\sqrt{5}}$.
Area of lateral surface =

$$\tfrac{1}{2} \times 10 \times 5 \times 3\sqrt{10 - 2\sqrt{5}} = 75\sqrt{10 - 2\sqrt{5}} = 176.33+.$$

$$\text{Apothem} = \sqrt{6^2 - (\tfrac{3}{2}\sqrt{10 - 2\sqrt{5}})^2} = \tfrac{3}{2}\sqrt{2(3 + \sqrt{5})}.$$

Area base =

$$\tfrac{1}{2} \times 5 \times 3\sqrt{10 - 2\sqrt{5}} \times \tfrac{3}{2}\sqrt{2(3 + \sqrt{5})} = \tfrac{45}{2}\sqrt{10 + 2\sqrt{5}}.$$

$$\text{Altitude} = \sqrt{(10)^2 - \left(\tfrac{3}{2}\sqrt{2(3 + \sqrt{5})}\right)^2} = \tfrac{1}{2}\sqrt{346 - 18\sqrt{5}}.$$

$$\therefore \text{Volume} = \tfrac{1}{3} \times \tfrac{1}{2} \times \tfrac{45}{2}\sqrt{(10 + 2\sqrt{5})(346 - 18\sqrt{5})} =$$

$$15\sqrt{205 + 32\sqrt{5}} = 249.45-.$$

3. Volume of cylinder $= 3.1416 \times 3^2 \times 6\frac{1}{2}$. Volume of cone $= 3.1416 \times 3^2 \times 1\frac{1}{6}$. Whole volume $= 3.1416 \times 9 \times 7\frac{2}{3}$ $= 3.1416 \times 69$. No. of quarts $= \dfrac{3.1416 \times 69 \times 4}{231} = 3.75+$.
Area of bottom $= 3.1416 \times 3^2$. Convex surface of cylinder $= 3.1416 \times 6 \times 6\frac{1}{2}$. Convex surface of cone $= 3.1416 \times 3 \times \sqrt{3^2 + (3\frac{1}{2})^2}$. Total surface $= 3.1416 \times 3 \times \{3 + 13 + \frac{1}{2}\sqrt{85}\} = 194.24$ sq. in.

4. Volumes are as $4^3 : 3^3 :: 64 : 27$.

5. Altitude of first $= \frac{3}{5} \times 2 = 1.2$. Volume of first $= \frac{1}{3} \times 3.1416 \times 4 \times 1.2 = 5.02656$. Volume of second $= \frac{27}{64} \times 5.02656 = 2.12058$.

6. The base of the triangle formed by altitude and slant height is $\frac{1}{2}(72 - 48) = 12$. $\therefore$ Slant height =

$$\sqrt{(60)^2 + (12)^2} = \sqrt{3744} = 12\sqrt{26}.$$

Lateral surface $= 4 \times \frac{1}{2}(48 + 72) \times 12\sqrt{26} = 14685.2-$.
Volume $= \frac{1}{3} \times 60 \times \{(48)^2 + (72)^2 + (48 \times 72)^2\} = 218880$.

OF THE SPHERE.

Page 237, 622.

1. The radius of the small circle is obviously $\sqrt{5^2-3^2}=4$. $2\pi r=2\times 4\times 3.1416=25.1328$.

3. Draw a circumference of a great circle, and on it lay off $AB=100°$; at A, by the method of (2), construct the angle $A=58°$; on the other side of the angle lay off $AC=80°$; pass an arc of a great circle through B and C.

4. As before, lay off $AB=75°$; at A construct $A=110°$; at B, $B=87°$, and produce the sides of the angles till they intersect at some point as C.

5. As before, lay off $AB=150°$; from B as a pole, with an arc equal to $BC=80°$, strike an indefinite arc; also, from A as a pole with an arc equal to $AC=100°$, strike another. If the intersection of these arcs be joined to A and B by arcs of great circles, we have the triangle desired. The second case is impossible, because $AC+BC<AB$, since $50°+85°<160°$.

7. $\dfrac{A+B+C-180°}{360°}\times 2\pi R^2=\dfrac{175}{360}\times 2\times 3.1416\times (10)^2=305.4$.

8. $\dfrac{A+B+C-180°}{360°}\times 2\pi R^2=\dfrac{100}{360}\times 2\times 3.1416\times 6^2=62.832$.

9. The same, since they have equal altitudes.

10. The areas are as 1578 : 2052 : 328, or as 789 : 1026 : 164.

11. $\dfrac{A+B+C-180°}{360°}\times 2\pi R^2=\dfrac{220}{360}\times 2\times 3.1416\times (3958)^2=60152356.2229$. $\dfrac{4\pi R^2}{8}=\frac{1}{2}\times 3.1416\times (3958)^2=24607782.0912$.

12. Construct the triangle as in (3). Then from the

vertices as poles strike arcs of great circles so as to intersect and form the polar triangle.

13. Same as latter part of (12).

14. In the polar $B'C' = 180° - 58° = 122°$, $A'C' = 180° - 67° = 113°$, $B' = 180° - 81° = 99°$.

15. $\frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.1416 \times 1 = 4.1888$. Taking the formula given in (18), $\pi \{r^2(b-a) - \frac{1}{3}(b^3 - a^3)\} = \pi \{1(\frac{3}{4} - \frac{1}{4}) - \frac{1}{3}[(\frac{3}{4})^3 - (\frac{1}{4})^3]\} = \pi \{\frac{1}{2} - \frac{1}{3}(\frac{13}{32})\} = \pi \times \frac{35}{96} = 1.1454$.

16. Surface of sphere $= 4\pi R^2$; of cylinder $= 2\pi R'H' = 2\pi R \times 2R = 4\pi R^2$; of cone $= \pi R''H'' = \pi \times R\sqrt{3} \times 2R\sqrt{3} = 6\pi R^2$. The entire surfaces are $4\pi R^2$, $6\pi R^2$, $9\pi R^2$.

17. Volume of sphere $= \frac{4}{3}\pi R^3$; of cylinder $= \pi R'^2 H' = \pi R^2 \times 2R = 2\pi R^3$; of cone $\frac{1}{3}\pi R''^2 H' = \frac{1}{3}\pi(R\sqrt{3})^2 \times 3R = 3\pi R^3$. Hence the entire surface of the cylinder is a mean proportional between that of the sphere and the cone. A similar proposition is true concerning the volumes.

18. From (**621**) and (**622**) we easily see that the required volume is

$$\begin{aligned}
&\pi\{(r-a)^2 r - \tfrac{1}{3}(r-a)^3\} - \pi\{(r-b)^2 r - \tfrac{1}{3}(r-b)^3\} \\
&= \pi\{r^3 - 2r^2 a + r a^2 - \tfrac{1}{3}(r^3 - 3r^2 a + 3ra^2 - a^3)\} \\
&- \pi\{r^3 - 2r^2 b + r b^2 - \tfrac{1}{3}(r^3 - 3r^2 b + 3rb^2 - b^3)\} \\
&= \pi\{r^2(b-a) - \tfrac{1}{3}(b^3 - a^3)\}.
\end{aligned}$$

KEY TO THE TRIGONOMETRY.

KEY TO THE TRIGONOMETRY.

PLANE TRIGONOMETRY.

[NOTE. — The full-faced figures in connection with the number of the page refer to the Articles in the Trigonometry. The numbers at the left hand, or above the Example, refer to the particular Example. Figures in the Trigonometry are denoted FIG. 23, etc.; Figures in the Key, FIG. 23 *a*, etc.]

DEFINITIONS AND FUNDAMENTAL RELATIONS.

Page 17, **47**.

1. $90° - 150° 21' 13''.5 = -60° 21' 13''.5$. $180° - 150° 21' 13''.5 = 29° 38' 46''.5$. $90° - 125° 15' = -35° 15'$. $180° - 125° 15' = 54° 45'$. $90° - 283° 21' 11'' = -193° 21' 11''$. $180° - 283° 21' 11'' = -103° 21' 11''$. $90° - 36° 5' 2'' = 53° 54' 58''$. $180° - 36° 5' 2'' = 143° 54' 58''$. $90° - 89° 0' 12'' = 59' 48''$. $180° - 89° 0' 12'' = 90° 59' 48''$.

2. $\frac{\pi}{3} = 60°$. $\frac{3}{2}\pi = 270°$. $2\pi = 360°$. $\frac{1}{4}\pi = 45°$. $1\frac{1}{4}\pi = 225°$.

3. $108° \div \frac{180°}{\pi} = \frac{108}{180}\pi = \frac{3}{5}\pi = \frac{3 \times 3.1416}{5} = 1.8849$. 2π.

$460^\circ \div \dfrac{180^\circ}{\pi} = \dfrac{460}{180}\pi = \dfrac{23}{9}\pi = \dfrac{23 \times 3.1416}{9} = 8.0285.$ $210^\circ \div \dfrac{180^\circ}{\pi} = \dfrac{210}{180}\pi = \tfrac{7}{6}\pi = \dfrac{7 \times 3.1416}{5} = 3.6652.$

4. $45^\circ = \tfrac{1}{4}\pi.$ $90^\circ = \tfrac{1}{2}\pi.$ $180^\circ = \pi.$ $225^\circ = 1\tfrac{1}{4}\pi.$ $1 = 57^\circ\ 17'\ 44''.8+.$

5. $1^\circ = \dfrac{18 \times 3.1416}{360} = .15708.$

6. $11'''\ 5^{\text{iv}} = \tfrac{665}{3600}$ of $1'' = 0''.18472+.$ $\therefore$ $12^\circ\ 22'\ 13''\ 11'''\ 5^{\text{iv}} = 12^\circ\ 22'\ 13''.18472+.$ $0^\circ.51 = 30'\ 36''.$ $\therefore$ $53^\circ.51 = 53^\circ\ 30'\ 36''.$

7. $2 \times 57^\circ\ 17'\ 44''.8 = 114^\circ\ 35'\ 29''.6.$ $3 \times 57^\circ\ 17'\ 44''.8 = 171^\circ\ 53'\ 14''.4.$ $27^\circ = 27^\circ \div \dfrac{180^\circ}{\pi} = \dfrac{27\pi}{180} = \tfrac{3}{20}\pi.$ $10^\circ = \dfrac{10^2 \times 3.1416 \times 2}{360} = 1.745.$ $1^\circ = \dfrac{2\pi R}{360} = 1.$ $\therefore$ $2\pi R = 360.$ $R = \dfrac{180}{\pi} = 57.29578-.$

12. To construct $\sec^{-1} 2$. Let OA be one side of the angle. With Oa, any convenient radius, strike a circumference. At a draw an indefinite tangent. From O as a centre, with a radius 2 Oa, strike arcs intersecting this tangent above and below. Join the intersections with the centre, and we have two angles, one in the first quadrant and one in the fourth, whose secant = 2. See FIG. 1 a.

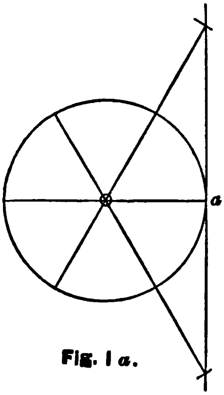

Fig. 1 *a*.

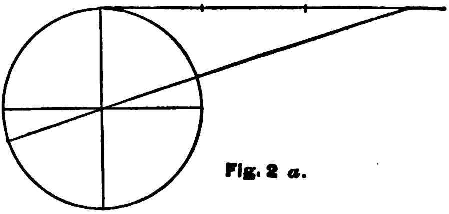

Fig. 2 *a*.

To construct $\cot^{-1} 3$. Having drawn a circumference and two diameters, one horizontal, the other vertical, at the upper extremity of the vertical draw a cotangent $= 3$ to the right. Join the end of this line to the centre, and produce the line to meet the circumference in the third quadrant. We thus have two angles, one in the first quadrant and one in the third, whose cotangent $= 3$. See FIG. 2*a*.

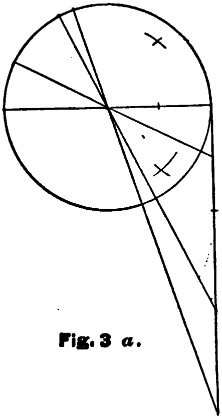
Fig. 3 *a*.

To construct $\operatorname{cosec}^{-1} 1\frac{1}{2}$. Draw the circumference and two diameters at right angles as before. At the upper extremity of the vertical, draw an indefinite cotangent. From the centre with a radius $1\frac{1}{2}$ **R**, strike arcs intersecting this cotangent at the right and left. Join these intersections with the centre, and we have two angles, one in the first quadrant and one in the second, whose cosecant $= 1\frac{1}{2}$. See FIG. 4*a*.

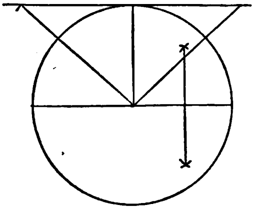
Fig. 4 *a*.

To construct $\tan^{-1} (-3)$. Draw the circumference and two diameters at right angles as before; at the right extremity of the horizontal diameter, draw a tangent, producing it downward till it $= 3$ **R.** Join the end of the tangent with the centre, and produce the line to meet the circumference in the second quadrant. Thus we have two angles, one in the second quadrant and one in the fourth, whose tangent $= -3$. See FIG. 3*a*.

13. To construct $\tan^{-1} 1$. Draw the tangent at the origin; lay off above a distance $=$ **R**; join the end of this line to the centre, and produce it to meet the circumference in the third quadrant. We thus have two angles, one in the

first quadrant and one in the second, whose tangent = 1. See FIG. 5*a*.

For $\tan^{-1}(-1)$, lay off a distance R below, and we shall obtain two angles, one in the second quadrant and one in the fourth, whose tangent = -1.

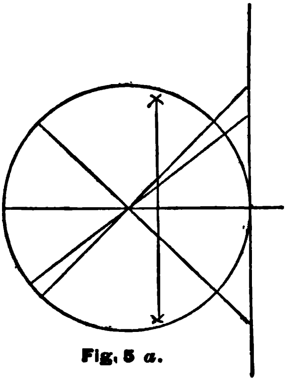

Fig. 5 *a*.

For $\tan^{-1}\frac{3}{4}$, make the radius of the circle four units of any convenient length, and on the indefinite tangent at the origin lay off above three of these units. See FIG. 5*a*.

$\tan^{-1}(-2)$ and $\tan^{-1}(-\frac{1}{2})$ need no further explanation. See FIG. 3*a*.

To construct $\cos^{-1}(-\frac{1}{3})$, make the radius three units of any convenient length. Lay off one of these units to the left of the centre, and through the point thus determined draw a perpendicular to the horizontal diameter. Join its intersections with the circumference to the centre, and we have two angles $\cos^{-1}(-\frac{1}{3})$, one in the second quadrant and one in the third.

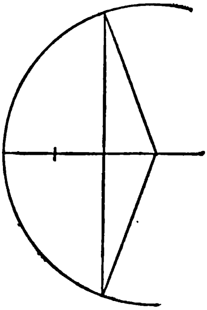

Fig. 6 *a*.

For $\sec^{-1}(-2)$ proceed as in $\sec^{-1} 2$ (12), save that we produce the lines through the centre, giving us two angles $\sec^{-1}(-2)$, one in the second quadrant and one in the third. See FIG. 1*a*.

For $\operatorname{cosec}^{-1}(-3)$ proceed as in $\operatorname{cosec}^{-1} 1\frac{1}{2}$ (12), save that we use a radius 3 R, and produce the lines through the centre, giving us two angles $\operatorname{cosec}^{-1}(-3)$, one in the third quadrant and one in the fourth. See FIG. 7*a*.

For $\operatorname{vers}^{-1}\frac{1}{4}$, with a radius equal to four units of any convenient length describe a circumference, draw the horizontal diameter, and from the origin toward the centre lay off one of these units. Through the point thus determined draw a perpendicular, meeting the circumference in the first and fourth quadrants. By joining these intersections with the centre we obtain two angles $\text{vers}^{-1}\frac{1}{4}$. See FIG. 8*a*.

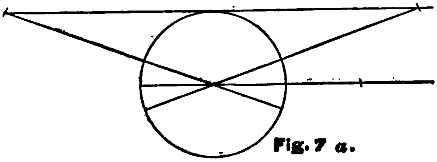
Fig. 7 *a*.

For $\text{vers}^{-1} 1\frac{1}{2}$, lay off on the horizontal diameter to the left of the centre two units, and draw a perpendicular as before, giving two angles $\text{vers}^{-1} 1\frac{1}{2}$, one in the second quadrant and one in the third.

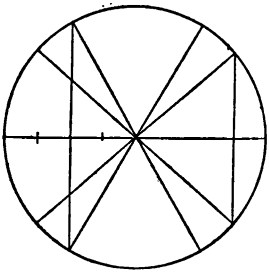
Fig. 8 *a*.

14. From $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = 1 - \cos^2 x$. $\therefore \sin x = \sqrt{1 - \cos^2 x}$. Also, $\cos^2 x = 1 - \sin^2 x$. $\therefore \cos x = \sqrt{1 - \sin^2 x}$. From $\cot x = \frac{1}{\tan x}$, $\tan x \cot x = 1$. $\tan x = \frac{\sin x}{\cos x}$. $\therefore \tan x \cos x = \sin x$, and $\cos x = \frac{\sin x}{\tan x}$. From $\cot x = \frac{\cos x}{\sin x}$, $\sin x = \frac{\cos x}{\cot x}$. From $\sin x = \frac{1}{\operatorname{cosec} x}$, $\sin x = \frac{1}{\sqrt{1 + \cot^2 x}}$. From $\cos x = \frac{1}{\sec x}$, $\cos x = \frac{1}{\sqrt{1 + \tan^2 x}}$. $\tan x = \frac{\sin x}{\cos x} = \sin x \sec x$. $\cot x = \frac{\cos x}{\sin x} = \cos x \operatorname{cosec} x$. $\tan x = \frac{\sin x}{\cos x} = \frac{1}{\operatorname{cosec} x} \div \frac{1}{\sec x} = \frac{\sec x}{\operatorname{cosec} x}$. $\sin x = \frac{\cos x}{\cot x} = \frac{\cos x}{\sqrt{\operatorname{cosec}^2 x - 1}}$. $\text{vers}\, x = 1 - \cos x = 1 - \frac{1}{\sec x} = \frac{\sec x - 1}{\sec x}$. $\text{vers}\, x = 1 - \cos x = 1 - \frac{\cot x}{\operatorname{cosec} x} = \frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x}$. $\cot x = \frac{\cos x}{\sin x} = \frac{\sqrt{1 - \sin^2 x}}{\sin x}$.

15. $\tan x = \frac{3}{4}$. $\sec x = \sqrt{1 + \tan^2 x} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$. $\cos x = \frac{1}{\sec x} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$. $\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$. $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$. $\cot x = \frac{1}{\tan x} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$. $\operatorname{vers} x = 1 - \cos x = 1 - \frac{4}{5} = \frac{1}{5}$. $\operatorname{covers} x = 1 - \sin x = 1 \frac{3}{5} = \frac{2}{5}$.

16. $\sin x = \frac{2}{3}$. $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{4}{9}} = \frac{1}{3}\sqrt{5}$. $\tan = \frac{\sin x}{\cos x} = \frac{2}{3} \div \frac{1}{3}\sqrt{5} = \frac{2}{5}\sqrt{5}$. $\cot x = \frac{\cos x}{\sin x} = \frac{1}{3}\sqrt{5} \div \frac{2}{3} = \frac{1}{2}\sqrt{5}$. $\operatorname{cosec} x = \frac{1}{\sin x} = 1 \div \frac{2}{3} = \frac{3}{2}$. $\sec x = \frac{1}{\cos x} = 1 \div \frac{1}{3}\sqrt{5} = \frac{3}{5}\sqrt{5}$. $\operatorname{vers} x = 1 - \cos x = 1 - \frac{1}{3}\sqrt{5}$. $\operatorname{covers} x = 1 - \sin x = 1 - \frac{2}{3} = \frac{1}{3}$.

17. $\sec x = 2$. $\tan x = \sqrt{\sec^2 x - 1} = \sqrt{3}$. $\cot x = \frac{1}{\tan x} = \frac{1}{3}\sqrt{3}$. $\cos x = \frac{1}{\sec x} = \frac{1}{2}$. $\sin x = \sqrt{1 - \cos^2 x} = \frac{1}{2}\sqrt{3}$. $\operatorname{vers} x = 1 - \cos x = \frac{1}{2}$. $\operatorname{covers} x = 1 - \sin x = 1 - \frac{1}{2}\sqrt{3}$. $\operatorname{cosec} x = \frac{1}{\sin x} = 1 \div \frac{1}{2}\sqrt{3} = \frac{2}{3}\sqrt{3}$.

18. $\tan x = -1$. $\sec x = \sqrt{1 + \tan^2 x} = \mp\sqrt{2}$. $\cot x = \frac{1}{\tan x} = \frac{1}{-1} = -1$. $\operatorname{cosec} x = \sqrt{1 + \cot^2 x} = \pm\sqrt{2}$. $\sin x = \frac{1}{\operatorname{cosec} x} = \pm\frac{1}{2}\sqrt{2}$. $\cos x = \frac{1}{\sec x} = \mp\frac{1}{2}\sqrt{2}$. $\operatorname{vers} x = 1 - \cos x = 1 \pm \frac{1}{2}\sqrt{2}$. $\operatorname{covers} x = 1 - \sin x = 1 \mp \frac{1}{2}\sqrt{2}$. We find $\tan^{-1}(-1)$ first in the second quadrant, and hence the upper sign of each ambiguous function will be the sign belonging to that function in the second quadrant.

19. For $\tan^{-1} 1$ and $\tan^{-1}(-1)$, see Ex. 13. To construct $\sin^{-1}(-\frac{1}{2})$, draw a circumference, the horizontal and vertical diameters. On the vertical diameter below the centre lay off a distance $\frac{1}{2}$ R. Through the point thus determined draw a perpendicular, producing it to meet the circumference

in the third and fourth quadrants, and we shall thus have an angle $\sin^{-1}(-\frac{1}{2})$ in each. See Fig. 9 *a*.

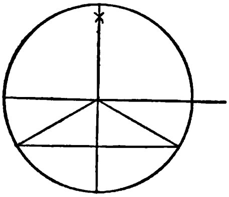

Fig. 9 *a*.

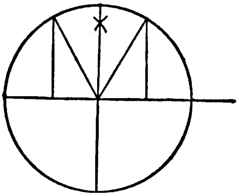

Fig. 10 *a*.

20. Two. In Fig. 10, produce PC till it meets the circumference in the second quadrant. If the intersection be joined to the centre we shall have a second angle $\sin^{-1}\frac{2}{3}$. For $\cos^{-1}\frac{1}{2}$ and $\cos^{-1}(-\frac{1}{2})$, see Fig. 10*a*. They are supplementary. For $\tan^{-1}3$ and $\tan^{-1}(-3)$, see Fig. 11*a*. They are supplementary.

21. $y = \sin^{-1}x$, $x = \sin y$. $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. $\tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1-x^2}}$. $\sec y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$. $\operatorname{cosec} y = \frac{1}{\sin y} = \frac{1}{x}$. $\therefore y = \operatorname{cosec}^{-1}\frac{1}{x}$. $\operatorname{vers} y = 1 - \cos y = 1 - \sqrt{1-x^2}$. $\cot y = \frac{1}{\tan y} = \frac{\sqrt{1-x^2}}{x}$. $\therefore y = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$.

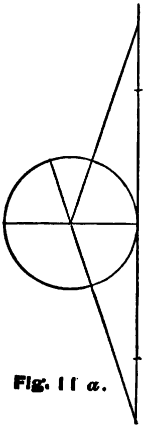

Fig. 11 *a*.

22. Each function of 450° = the same function of 450° − 360°, or 90°. Each function of 1350° = the same function of 1350° − 3 × 360°, or 270°. Each function of 900° = the same function of 900° − 2 × 360°, or 180°.

23. $4n\frac{\pi}{2}$ is any even multiple of 180°. $\therefore \sin 4n\frac{\pi}{2} = 0$.

$(4n+1)\frac{\pi}{2}$ represents 90° increased by every even multiple of 180°. $\therefore \sin(4n+1)\frac{\pi}{2}=1$. $(4n+2)\frac{\pi}{2}$ is any odd multiple of 180°. $\therefore \sin(4n+2)\frac{\pi}{2}=0$. $(4n+3)\frac{\pi}{2}$ represents 270° increased by any even multiple of 180°.

$$\therefore \sin(4n+3)\frac{\pi}{2}=-1.$$

The others may be shown in a similar way.

24. $-110°$ ends in the third quadrant; $-35°$ in the fourth; $-500° = -360° - 140°$ in the third; $-2000° = -5 \times 360° - 200°$ ends in the second. The signs of the functions will then be the same as of angles terminating in the third, the fourth, the third, and the second quadrants, respectively.

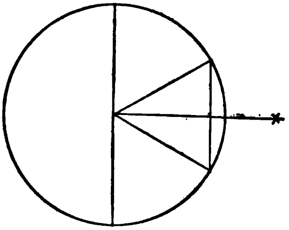

Fig. 12 a.

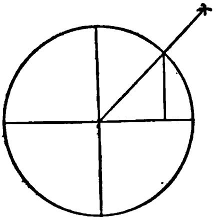

Fig. 13 a.

25. The sine of 30°, being the perpendicular let fall from the termination of the arc upon the diameter through the origin, is seen at once to be one-half the chord of 60°, or $\frac{1}{2}$. $\cos 30° = \sqrt{1-\sin^2 30°} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$. Applying the fundamental relations, we obtain $\tan x = \frac{1}{3}\sqrt{3}$, $\cot x = \sqrt{3}$, $\sec x = \frac{2}{3}\sqrt{3}$, $\operatorname{cosec} x = 2$. See FIG. 12 *a*.

26. The triangle formed by the sine and cosine of 45° with the radius is obviously isosceles. $\therefore \sin 45° = \cos 45°$. $\therefore \sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$. We easily get $\tan x = \cot x = 1$, $\sec x = \operatorname{cosec} x = \sqrt{2}$. See FIG. 13 *a*.

FUNCTIONS OF THE SUM OR DIFFERENCE OF ANGLES.

Page 26, 57.

1. In FIG. (*a*) the description is obvious. $PD = \sin x$, $OD = \cos x$, $P'E = \sin y$, $OE = \cos y$, $P'D' = \sin(x+y)$. Now $P'D' = P'L + LD' = P'L + EF$. By similar triangles, $\frac{EF}{OE} = \frac{PD}{OP}$, or $\frac{EF}{\cos y} = \frac{\sin x}{1}$. $\therefore EF = \sin x \cos y$. Also $\frac{P'L}{P'E} = \frac{OD}{OP}$, or $\frac{P'L}{\sin y} = \frac{\cos x}{1}$. $\therefore P'L = \cos x \sin y$. $\therefore \sin(x+y) = \sin x \cos y + \cos x \sin y$.

2 and 3. Explanations in book.

4. In FIG. (*a*) $\cos(x+y) = \sin x \sin y - \cos x \cos y$, but $\cos(x+y)$ is $-$; making the change, $\cos(x+y) = \cos x \cos y - \sin x \sin y$. In FIG. (*b*) $\cos(x+y) = \cos x \cos y + \sin x \sin y$, but $\cos y$ and $\cos(x+y)$ are $-$; making the changes, $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

In FIG. (*c*) $\cos(x+y) = \cos x \cos y + \sin x \sin y$, but $\cos x$ and $\cos(x+y)$ are $-$; making the changes, $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

5 and 6. In FIG. (*d*) the proportions are the same as in FIG's. (*a*), (*b*), (*c*), and need no further explanation.

7. $\sin 75^\circ = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2} \times \frac{1}{2} = \frac{1}{4}(\sqrt{6} + \sqrt{2}) = \frac{1}{4}(2.44948974 + 1.41421356) = .9659258$. Similarly, $\sin 15^\circ = \frac{1}{4}(\sqrt{6} - \sqrt{2}) = .2588190$. $\tan 75^\circ = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{3}\sqrt{3}}{1 - \frac{1}{3}\sqrt{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = 2 + \sqrt{3} = 3.7320508$. $\tan 15^\circ = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = 2 - \sqrt{3} = .2679492$.

8. $\sin 15^\circ = \sqrt{\frac{1}{2}(1 - \cos 30^\circ)} = \sqrt{\frac{1}{2}(1 - \frac{1}{2}\sqrt{3})} =$

$\frac{1}{2}\sqrt{2-\sqrt{3}} = .2588190.$ cos 15° $= \sqrt{\frac{1}{2}(1+\cos 30°)} = \sqrt{\frac{1}{2}(1+\frac{1}{2}\sqrt{3})} = \frac{1}{2}\sqrt{2+\sqrt{3}} = .9659258.$ tan 15° $= \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} = 2-\sqrt{3} = .2679492.$ cot 15° $= \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2 + \sqrt{3} = 3.7320508.$ sin 7° 30′ $= \sqrt{\frac{1}{2}(1-\cos 15°)} = \sqrt{\frac{1}{2}(1-\frac{1}{2}\sqrt{2+\sqrt{3}})} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}} = .1305262.$ cos 7° 30′ $= \sqrt{\frac{1}{2}(1+\cos 15°)} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}} =$ .9914449, etc., etc.

9. Of 45° divided by any multiple of 2.

10. $\sin(x+y+z) = \sin\{(x+y)+z\} = \sin(x+y)\cos z + \cos(x+y)\sin z = (\sin x\cos y + \cos x\sin y)\cos z + (\cos x\cos y - \sin x\sin y)\sin z = \sin x\cos y\cos z + \cos x\sin y\cos z + \cos x\cos y\sin z - \sin x\sin y\sin z.$

$\cos(x+y+z) = \cos\{(x+y)+z\} = \cos(x+y)\cos z - \sin(x+y)\sin z = (\cos x\cos y - \sin x\sin y)\cos z - (\sin x\cos y + \cos x\sin y)\sin z = \cos x\cos y\cos z - \sin x\sin y\cos z$ $\sin x\cos y\sin z - \cos x\sin y\sin z.$

$\tan(x+y+z) = \tan\{(x+y)+z\} =$

$$\frac{\dfrac{\tan x+\tan y}{1-\tan x\tan y}+\tan z}{1-\dfrac{\tan x+\tan y}{1-\tan x\tan y}\tan z} = \frac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan x\tan z-\tan y\tan z}.$$

$\cot(x+y+z) = \cot\{(x+y)+z\} =$

$$\frac{\dfrac{\cot x\cot y-1}{\cot x+\cot y}\cot z-1}{\dfrac{\cot x\cot y-1}{\cot x+\cot y}+\cot z} = \frac{\cot x\cot y\cot z-\cot x-\cot y-\cot z}{\cot x\cot y+\cot x\cot z+\cot y\cot z-1}.$$

11. $\sec(x+y) = \dfrac{1}{\cos(x+y)} = \dfrac{1}{\cos x \cos y - \sin x \sin y}$

$$= \frac{1}{\dfrac{1}{\sec x \sec y} - \dfrac{1}{\operatorname{cosec} x \operatorname{cosec} y}} = \frac{\sec x \sec y \operatorname{cosec} x \operatorname{cosec} y}{\operatorname{cosec} x \operatorname{cosec} y - \sec x \sec y}.$$

12. $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x = (2\cos^2 x - 1)\cos x - 2(1 - \cos^2 x)\cos x = 4\cos^3 x - 3\cos x.$

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} =$$

$$\frac{\dfrac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \dfrac{2\tan x}{1 - \tan^2 x}\tan x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

$$\cot 3x = \cot(2x + x) = \frac{\cot 2x \cot x - 1}{\cot 2x + \cot x} =$$

$$\frac{\dfrac{\cot^2 x - 1}{2\cot x}\cot x - 1}{\dfrac{\cot^2 x - 1}{2\cot x} + \cot x} = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}.$$

13. $\sin 4x = \sin\{2(2x)\} = 2\sin 2x \cos 2x = 2(2\sin x \cos x)(1 - 2\sin^2 x) = 4(\sin x - 2\sin^3 x)\cos x.$

15. $\tan \frac{1}{2}x = \dfrac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} = \dfrac{2\sin^2 \frac{1}{2}x}{2\sin \frac{1}{2}x \cos \frac{1}{2}x} = \dfrac{1 - \cos x}{\sin x}.$

$\cot \frac{1}{2}x = \dfrac{\cos \frac{1}{2}x}{\sin \frac{1}{2}x} = \dfrac{2\cos^2 \frac{1}{2}x}{2\sin \frac{1}{2}x \cos \frac{1}{2}x} = \dfrac{1 + \cos x}{\sin x}.$

16. $\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1) = .3090170.$ $\cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}} = .9510565.$ $\tan 18^\circ = \dfrac{\sqrt{5} - 1}{\sqrt{10 + 2\sqrt{5}}} = \sqrt{\dfrac{6 - 2\sqrt{5}}{10 + 2\sqrt{5}}} = \sqrt{\tfrac{1}{5}(5 - 2\sqrt{5})} = .3249197.$ $\cot 18^\circ = \dfrac{\sqrt{10 + 2\sqrt{5}}}{\sqrt{5} - 1} = \sqrt{5 + 2\sqrt{5}} = 3.0776835.$

17. $\sin 3^\circ = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ =$

$$\tfrac{1}{4}(\sqrt{5}-1)\left\{\tfrac{1}{4}(\sqrt{6}+\sqrt{2})\right\}-\tfrac{1}{4}\sqrt{10+2\sqrt{5}}\left\{\tfrac{1}{4}(\sqrt{6}-\sqrt{2})\right\}$$

$$=\frac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1)-\frac{\sqrt{3}-1}{8}\sqrt{5+\sqrt{5}}=.0523360.$$

$\cos 3^\circ = \cos 18^\circ \cos 15^\circ + \sin 18^\circ \sin 15^\circ = \tfrac{1}{4}\sqrt{10+2\sqrt{5}}$

$$\left\{\tfrac{1}{4}(\sqrt{6}+\sqrt{2})\right\}+\tfrac{1}{4}(\sqrt{5}-1)\left\{\tfrac{1}{4}(\sqrt{6}-\sqrt{2})\right\}=\frac{\sqrt{3}-1}{8\sqrt{2}}$$

$$(\sqrt{5}-1)+\frac{\sqrt{3}+1}{8}\sqrt{5+\sqrt{5}}=.9986295.$$

$$\tan 3^\circ=\frac{\dfrac{\sqrt{3}+1}{8\sqrt{2}}(\sqrt{5}-1)-\dfrac{\sqrt{3}-1}{8}\sqrt{5+\sqrt{5}}}{\dfrac{\sqrt{3}-1}{8\sqrt{2}}(\sqrt{5}-1)+\dfrac{\sqrt{3}+1}{8}\sqrt{5+\sqrt{5}}}=$$

$$\frac{\sqrt{10-2\sqrt{5}}-2}{1+2\sqrt{3}+\sqrt{5}}=.0524078.$$

$$\cot 3^\circ=\frac{1+2\sqrt{3}+\sqrt{5}}{\sqrt{10-2\sqrt{5}}-2}=19.081137.$$

By using $\sin 2x = 2 \sin x \cos x$, and $\cos 2x = \cos^2 x - \sin^2 x$, we find $\sin 6^\circ = -\dfrac{1}{8}(\sqrt{5}+1)+\dfrac{\sqrt{3}}{4\sqrt{2}}\sqrt{5-\sqrt{5}}=.1045285,$

$$\cos 6^\circ=\frac{\sqrt{3}}{8}(\sqrt{5}+1)+\frac{1}{4\sqrt{2}}\sqrt{5-\sqrt{5}}=.9945219,$$

$$\sin 12^\circ=-\frac{\sqrt{3}}{8}(\sqrt{5}-1)+\frac{1}{4\sqrt{2}}\sqrt{5+\sqrt{5}}=.2079117,$$

$$\cos 12^\circ=\frac{1}{8}(\sqrt{5}-1)+\frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5+\sqrt{5}}=.9781476,$$

$$\sin 24^\circ=\frac{\sqrt{3}}{8}(\sqrt{5}+1)-\frac{1}{4\sqrt{2}}\sqrt{5-\sqrt{5}}=.4067366,$$

$$\cos 24^\circ=\frac{1}{8}(\sqrt{5}+1)+\frac{\sqrt{3}}{4\sqrt{2}}\sqrt{5-\sqrt{5}}=.9135454.$$

FORMULÆ FOR RENDERING CALCULABLE BY LOGARITHMS THE ALGEBRAIC SUM OF TRIGONOMETRICAL FUNCTIONS.

Page 30, 62.

1. $\cot x + \cot y = \frac{\cos x}{\sin x} + \frac{\cos y}{\sin y} = \frac{\cos x \sin y + \sin x \cos y}{\sin x \sin y} = \frac{\sin (x+y)}{\sin x \sin y}.$

2. $\sec x + \sec y = \frac{1}{\cos x} + \frac{1}{\cos y} = \frac{\cos y + \cos x}{\cos x \cos y} = \frac{2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)}{\cos x \cos y}.$

3. $\sec x - \sec y = \frac{1}{\cos x} - \frac{1}{\cos y} = \frac{\cos y - \cos x}{\cos x \cos y} = \frac{2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)}{\cos x \cos y}.$

4 and 5. See (57.)

6. $\frac{\sin x + \sin y}{\cos x + \cos y} = \frac{2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)}{2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)} = \tan \frac{1}{2}(x+y).$

7. $\frac{\sin x - \sin y}{\cos x + \cos y} = \frac{2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)}{2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)} = \tan \frac{1}{2}(x-y).$

8. $\frac{\sin x + \sin y}{\cos x - \cos y} = \frac{2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)}{-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)} = -\cot \frac{1}{2}(x-y).$

9. $\frac{\sin x - \sin y}{\cos x - \cos y} = \frac{2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)}{-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)} = -\cot \frac{1}{2}(x+y).$

CONSTRUCTION AND USE OF TRIGONOMETRICAL TABLES.

Page 35, 70.

4. $\cos 42^\circ\ 27' = .73787$. $\cos 42^\circ\ 28' = .73767$. $.73787 - \frac{1}{5} \times 20 = .73783 = \cos 42^\circ\ 27'\ 12''$. $\tan 42^\circ\ 27' = .914727$. $\tan 42^\circ\ 28' = .915261$. $.914727 + \frac{1}{5} \times 534 = .914834 = \tan 42^\circ\ 27'\ 12''$. $\cot 42^\circ\ 27' = 1.09322$. $\cot 42^\circ\ 28' = 1.09258$. $1.09322 - \frac{1}{5} \times 64 = 1.09309 = \cot 42^\circ\ 27'\ 12''$.

5. $\cos 151^\circ\ 23' = \sin (151^\circ\ 23' - 90^\circ) = \sin 61^\circ\ 23' = .877844$. $\tan 132^\circ\ 36' = \cot (132^\circ\ 36' - 90^\circ) = \cot 42^\circ\ 36' = 1.08749$. $\cot 116^\circ\ 7' = \tan (116^\circ\ 7' - 90^\circ) = \tan 26^\circ\ 7' = .490256$.

7. $\log \sin 24^\circ\ 28' = 9.617172$. $9.617172 - 26 \times 4.62 = 9.617052 = \log \sin 24^\circ\ 27'\ 34''$. $\log \cos 26^\circ\ 12'\ 20'' = 9.952918 - 20 \times 1.04 = 9.952897$. $\log \tan 26^\circ\ 12'\ 20'' = 9.692019 + 20 \times 5.31 = 9.692125$. $\log \cot 126^\circ\ 23'\ 50'' = \log \cot (180^\circ - 126^\circ\ 23'\ 50'') = \log \cot 53^\circ\ 36'\ 10'' = 9.867623 - 10 \times 4.41 = 9.867579$.

9. $\sin^{-1} .62592 = 38^\circ\ 45'$. $\sin 38^\circ\ 45' - \sin 38^\circ\ 44' = 22$. $38^\circ\ 45' - \frac{9}{22} \times 60'' = 38^\circ\ 44'\ 35''$, etc.

$\cos^{-1} .34257 = 69^\circ\ 58'$. $\cos 69^\circ\ 57' - \cos 69^\circ\ 58' = 27$. $69^\circ\ 58' - \frac{11}{27} \times 60'' = 69^\circ\ 57'\ 36''$, etc.

$\tan^{-1} .468434 = 25^\circ\ 6'$. $\tan^{-1} .468531 = 25^\circ\ 6' + \dfrac{97''}{5.95} = 25^\circ\ 6'\ 16''$, etc. $\cot^{-1} .876462 = 48^\circ\ 46'$. $\cot^{-1} .876434 = 48^\circ\ 46' + \dfrac{28''}{8.64} = 48^\circ\ 46'\ 3''$, etc.

10. $\sin^{-1} .34257 = 20^\circ\ 2'$. $\sin 20^\circ\ 2' - \sin 20^\circ\ 1' = 28$. $20^\circ\ 2' - \frac{1}{28} \times 60'' = 20^\circ\ 1'\ 58''$. $180^\circ + 20^\circ\ 1'\ 58'' = 200^\circ\ 1'\ 58''$, etc. $= \sin^{-1} (-.34256)$. $\sin^{-1} .62592 = 38^\circ\ 45'$. $\sin 38^\circ\ 45' - \sin 38^\circ\ 44' = 22$. $38^\circ\ 45' - \frac{8}{22} \times 60'' = 38^\circ\ 44'\ 38'' = \sin^{-1} .62584$. $\cos^{-1} (-.62584) = 90^\circ + 38^\circ\ 44'\ 38''$

$= 128^\circ\ 41'\ 38''$, etc. $\tan^{-1} 3.41604 = 73^\circ\ 41'$. $73^\circ\ 41' + \frac{17''}{6.01} = 73^\circ\ 41'\ 3''$. $\tan^{-1}(-3.41621) = 180^\circ - 73^\circ\ 41'\ 3'' = 106^\circ\ 18'\ 57''$, etc.

12. $\log \cos^{-1} 9.778119 = 53^\circ\ 8'$. $53^\circ\ 8' - \frac{32''}{2.81} = 53^\circ\ 7'\ 49''$. $360^\circ - 53^\circ\ 7'\ 49'' = 306^\circ\ 52'\ 11''$.

13. $\log \cos^{-1} 9.246775 = 79^\circ\ 50'$. $79^\circ\ 50' - \frac{56''}{11.73} = 79^\circ\ 49'\ 55''$. $360^\circ - 79^\circ\ 49'\ 55'' = 280^\circ\ 10'\ 5''$. $\log \cos^{-1} 9.889374 = 39^\circ\ 11'$. $39^\circ\ 11' + \frac{2''}{1.72} = 39^\circ\ 11'\ 1''$. $360^\circ - 39^\circ\ 11'\ 1'' = 320^\circ\ 48'\ 59''$.

14. $\log \tan^{-1} 9.895672 = 38^\circ\ 11'$. $38^\circ\ 11' + \frac{88''}{4.33} = 38^\circ\ 11'\ 20''$. $\log \tan^{-1} 10.531186 = 73^\circ\ 36'$. $73^\circ\ 36' - \frac{132''}{7.77} = 73^\circ\ 35'\ 43''$. $\log \tan^{-1} 11.215592 = 86^\circ\ 31'$. $86^\circ\ 31' + \frac{721''}{34.64} = 86^\circ\ 31'\ 21''$.

15. $\log \cot^{-1} 10.008591 = 44^\circ\ 26'$. $44^\circ\ 26' - \frac{97''}{4.21} = 44^\circ\ 25'\ 37''$. $\log \cot^{-1} 9.638302 = 66^\circ\ 30'$. $66^\circ\ 30' - \frac{34''}{5.76} = 66^\circ\ 29'\ 54''$. $\log \cot^{-1} 9.436570 = 74^\circ\ 43'$. $74^\circ\ 43' - \frac{241''}{8.28} = 74^\circ\ 42'\ 31''$.

17. $\log \sin^{-1} 9.653808 = 26^\circ\ 47'$. $26^\circ\ 47' + \frac{17''}{4.17} = 26^\circ\ 47'\ 4''$. Now $\cos(90 + x') = -\sin x'$. $\therefore x = 90^\circ + 26^\circ\ 47'\ 4'' = 116^\circ\ 27'\ 4''$.

18. $\log \cot^{-1} 9.884196 = 52^\circ\ 33'$. $52^\circ\ 33' + \frac{66''}{4.36} = 52^\circ\ 33'\ 15''$. Now $\tan(90^\circ + x') = -\cot x'$. $\therefore x = 90^\circ + 52^\circ\ 33'\ 15'' = 142^\circ\ 33'\ 15''$.

19. None.

20. $\log \tan^{-1} 9.564202 = 20^\circ\ 8'$. $20^\circ\ 8' + \frac{97''}{6.57} = 20^\circ\ 8'\ 15''$. Now $\cot\,(90^\circ + x') = -\tan x'$. $\therefore\ x = 90^\circ + 20^\circ\ 8'\ 15'' = 110^\circ\ 8'\ 15''$.

21. $\log \sin^{-1} 9.562468 = 21^\circ\ 25'$. $\therefore\ x = 90^\circ + 21^\circ\ 25' = 111^\circ\ 25'$. $\log \cot^{-1} 10.764141 = 9^\circ\ 46'$. $9^\circ\ 46' - \frac{74''}{12.58} = 9^\circ\ 45'\ 54''$. $\therefore\ x = 90^\circ + 9^\circ\ 45'\ 54'' = 99^\circ\ 45'\ 54''$. $\log\,(-\sin x) = 8.886432$ is impossible when $x < 180^\circ$. $\log \tan^{-1} 11.151740 = 85^\circ\ 58'$. $85^\circ\ 58' + \frac{421''}{29.95} = 85^\circ\ 58'\ 14''$. $\therefore\ x = 90^\circ + 85^\circ\ 58'\ 14'' = 175^\circ\ 58'\ 14''$.

22. $10 + \log 2.16145 = 10.334745$. $\log \tan^{-1} 10.334634 = 65^\circ\ 10'$. $\log \tan^{-1} 10.334745 = 65^\circ\ 10' + \frac{111''}{5.52} = 65^\circ\ 10'\ 20''$.

23. $\log \cos 35^\circ\ 23' - 10 = \bar{1}.911315$. $\log^{-1} \bar{1}.911315 = .57904 = \cos 35^\circ\ 23'$. $\tan^{-1} 2 = \log \tan^{-1}\,(10 + \log 2) = \log \tan^{-1} 10.301030$. $\log \tan^{-1} 10.300999 = 63^\circ\ 26'$. $\therefore\ \log \tan^{-1} 10.301030 = 63^\circ\ 26' + \frac{31''}{5.26} = 63^\circ\ 26'\ 6''$.

24. $\log 1.82645 + 10 = 10.261608$. $\log \cot^{-1} 10.261629 = 28^\circ\ 42'$. $\therefore\ \log \cot^{-1} 10.261608 = 28^\circ\ 42' + \frac{21''}{5.00} = 28^\circ\ 42'\ 4''$. $\therefore\ x = 90^\circ + 28^\circ\ 42'\ 4'' = 118^\circ\ 42'\ 4''$. $\log .42536 + 10 = 9.628757$. $\log \sin^{-1} 9.628647 = 25^\circ\ 10'$. $\therefore\ \log \sin 9.628757 = 25^\circ\ 10' + \frac{110''}{4.48} = 25^\circ\ 10'\ 25''$. $\therefore\ x = 90^\circ + 25^\circ\ 10'\ 25'' = 115^\circ\ 10'\ 25''$.

25. $\tan x = \frac{\sin x}{\cos x}$. $\therefore\ \log \tan x = \log \sin x - \log \cos x$. Now the tabular logarithmic trigonometrical functions are all 10 too great. But if we subtract tabular logarithmic cosine from tabular logarithmic sine the 10's are removed, and hence

we shall have to add 10 to their difference to obtain the tabular logarithmic tangent. Similarly for logarithmic cotangent. Since $\tan x \cot x = 1$, $\log \tan x + \log \cot x = 0$, or tabular log tan + tabular log cot $= 10 + 10 = 20$. Since the sum of tabular log tan and tabular log cot is always 20, any increase in one will cause the same decrease in the other, and hence one column of tabular differences will be sufficient.

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2. $\log \sin 56' 26'' = 4.685575 + \log 3386 - .000020 = 8.215242$. $\log \tan 56' 26'' = 4.685575 + \log 3386 + .000039 = 8.215301$. $\log \cot 56' 26'' = 20 - \log \tan 56' 26'' = 20 - 8.215301 = 11.784699$.

3. $\log \cos 88° 17' 44'' = \log \sin 1° 42' 16'' = 4.685575 + \log 6136 - .000064 = 8.473396$. $\log \tan 88° 17' 44'' = 20 - \log \tan 1° 42' 16'' = 20 - (4.685575 + \log 6136 + .000128) = 20 - 8.473588 = 11.526412$. $\log \cot 88° 17' 44'' = 8.473588$.

6. $\log \cot^{-1} 12.197148 = \log \tan^{-1} 7.802852 = 21' 49''.8$. $\log x = \log \tan x'' - 4.685575 - \text{Diff.} = 7.802852 - 4.685575 - .000006$. $\therefore\ x'' = 1310''$. $\log \cot^{-1} 12.197148 = 21' 50''$.

TRIGONOMETRICAL SOLUTION OF PLANE TRIANGLES.

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1. Referring to FIG. 12, $\sin B = \cos C = \frac{3}{5}$, $\operatorname{cosec} B = \sec C = \frac{5}{3}$, $\tan B = \cot C = \frac{3}{4}$, $\cot B = \tan C = \frac{4}{3}$, $\cos B = \sin C = \frac{4}{5}$, $\sec B = \operatorname{cosec} C = \frac{5}{4}$. The functions are the same when the sides are 6, 8, and 10.

3. $B = \tan^{-1} 2 = \sec^{-1} \sqrt{1 + 2^2} = \sec^{-1} \sqrt{5} = \cos^{-1} \frac{1}{5}\sqrt{5}$. $\cos C = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{1}{5}} = \frac{2}{5}\sqrt{5}$. $\therefore\ C = \cos^{-1} \frac{2}{5}\sqrt{5}$. $b = a \cos C = 12 \times \frac{2}{5}\sqrt{5} = \frac{24}{5}\sqrt{5}$. $c = a \cos B = 12 \times \frac{1}{5}\sqrt{5} = \frac{12}{5}\sqrt{5}$.

4. $b = 20$, $c = 32$. $\tan \mathrm{B} = \dfrac{b}{c} = \frac{5}{8}$. $\tan \mathrm{C} = \frac{8}{5}$. $\sec \mathrm{C} = \sqrt{1 + \tan^2 \mathrm{C}} = \sqrt{1 + \frac{64}{25}} = \frac{1}{5}\sqrt{89}$. $\therefore$ $a = b \sec \mathrm{C} = 20 \times \frac{1}{5}\sqrt{89} = 4\sqrt{89}$.

5. $a = 120$, $b = 100$. $\sin \mathrm{B} = \frac{100}{120} = \frac{5}{6}$. $\cos \mathrm{B} = \sqrt{1 - (\frac{5}{6})^2} = \frac{1}{6}\sqrt{11}$. $\therefore$ $\tan \mathrm{B} = \frac{5}{6} \div \frac{1}{6}\sqrt{11} = \frac{5}{11}\sqrt{11}$. $\cot \mathrm{C} = \frac{5}{11}\sqrt{11}$. $\therefore$ $\operatorname{cosec} \mathrm{C} = \sqrt{1 + \frac{25}{11}} = \frac{6}{11}\sqrt{11}$. $c = a \div \operatorname{cosec} \mathrm{C} = 120 \div \frac{6}{11}\sqrt{11} = 20\sqrt{11}$.

EXAMPLES.

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2. $\mathrm{AC} = b = 94.6$, $\mathrm{C} = 56° 30'$. $\mathrm{A} = 90° - 56° 30' = 33° 30'$. $a = b \sin \mathrm{A} = 94.6 \sin 33° 30' = 94.6 \times .55194 = 52.21$. $c = b \sin \mathrm{C} = 94.6 \sin 56° 30' = 94.6 \times .83389 = 78.89$.

3. $\mathrm{BF} = d = 127.9$, $\mathrm{B} = 40° 10' 30''$. $\mathrm{F} = 90° - 40° 10' 30'' = 49° 49' 30''$. $b = d \sin \mathrm{B} = 127.9 \times \sin 40° 10' 30'' = 127.9 \times .64512 = 82.51$. $f = d \cos \mathrm{B} = 127.9 \times \cos 40° 10' 30'' = 127.9 \times .76408 = 97.73$.

4. $\mathrm{DE} = c = 75$, $\mathrm{CE} = d = 50.59$. $\tan \mathrm{D} = d \div c = 50.59 \div 75 = .674533$. $\therefore$ $\mathrm{D} = 34° 0' 3''$. $\mathrm{C} = 90° - \mathrm{D} = 90° - 34° 0' 3'' = 55° 59' 57''$. $\mathrm{E} = d \div \sin 34° 0' 3'' = 50.59 \div .5592 = 90.47$.

5. $\mathrm{CD} = e = 264$, $\mathrm{CE} = d = 135.97$. $\sin \mathrm{D} = d \div e = 135.97 \div 264 = .51504$. $\therefore$ $\mathrm{D} = 31°$. $\mathrm{C} = 90° - 31° = 59°$. $c = e \sin \mathrm{C} = 264 \times \sin 59° = 264 \times .85717 = 226.29$.

6. $\mathrm{BC} = a = 435$, $\mathrm{B} = 44°$. $\mathrm{C} = 90° - 44° = 46°$. $b = a \sin \mathrm{B} = 435 \times \sin 44° = 435 \times .69466 = 302.18$. $c = a \cos \mathrm{B} = 435 \times \cos 44° = 435 \times .71934 = 312.91$.

7. $\mathrm{BC} = a = 64$, $\mathrm{CA} = b = 51.778$. $\sin \mathrm{B} = b \div a = 51.778 \div 64 = .80903$. $\therefore$ $\mathrm{B} = 54° 0' 3''$. $\mathrm{C} = 90° - 54° 0' 3'' = 35° 59' 57''$. $c = a \sin \mathrm{C} = 64 \times \sin 35° 59' 57'' = 64 \times .58778 = 37.62$.

8. $\mathrm{BC} = a = 749$, $\mathrm{AC} = b = 548.255$. $\sin \mathrm{B} = b \div a =$

$548.255 \div 749 = .73185$. $\therefore$ $B = 47^\circ\ 3'\ 9''$. $C = 90^\circ - 47^\circ\ 3'\ 9'' = 42^\circ\ 56'\ 51''$. $c = a \cos B = 749 \times \cos 47^\circ\ 3'\ 9'' = 749 \times .68133 = 510.31$.

9. $a = 125.7$, $B = 75^\circ\ 12'\ 23''$. $C = 90^\circ - 75^\circ\ 12'\ 23'' = 14^\circ\ 47'\ 37''$. $c = a \sin C = 125.7 \times \sin 14^\circ\ 47'\ 37'' = 125.7 \times .25534 = 32.096$. $b = a \cos C = 125.7 \times \cos 14^\circ\ 47'\ 37'' = 125.7 \times .96685 = 121.53$.

10. $b = 388.875$, $C = 27^\circ\ 38'\ 50''$. $B = 90^\circ - 27^\circ\ 38'\ 50'' = 62^\circ\ 21'\ 10''$. $c = b \tan C = 388.875 \times \sin 27^\circ\ 38'\ 50'' = 388.875 \times .523837 = 203.7$. $a = b \div \cos C = 388.875 \div \cos 27^\circ\ 38'\ 50'' = 388.875 \div .88582 = 439$.

12. $AC = b = 340$, $AB = c = 200$. $\sin C = c \div b$. $\therefore$ $\log \sin C = \log c - \log b + 10 = 9.769551$. $\therefore$ $C = 36^\circ\ 1'\ 54''$. $A = 90^\circ - 36^\circ\ 1'\ 54'' = 53^\circ\ 58'\ 6''$. $a = b \sin A$. $\therefore$ $\log a = \log b + \log \sin A - 10 = 2.439262$. $\therefore$ $a = 274.95$.

13. $AB = c = 736.3$, $BC = a = 500$. $\tan C = c \div a$. $\therefore$ $\log \tan C = \log c - \log a + 10 = 10.168085$. $\therefore$ $C = 55^\circ\ 49'\ 15''$. $B = 90^\circ - 55^\circ\ 49'\ 15'' = 34^\circ\ 10'\ 45''$. $b = c \div \sin C$. $\therefore$ $\log b = 2.949400$. $\therefore$ $b = 890.02$.

14. $BD = f = 246.32$, $DF = b = 380.07$. $\tan F = f \div b$. $\therefore$ $\log \tan F = \log f - \log b + 10 = 9.811636$. $\therefore$ $F = 32^\circ\ 56'\ 49''$. $B = 90^\circ - 32^\circ\ 56'\ 49'' = 57^\circ\ 3'\ 11''$. $d = f \div \cos B$. $\therefore$ $\log d = \log f - \log \cos B + 10 = 2.656011$. $\therefore$ $d = 452.91$.

15. $AB = c = 249$, $A = 29^\circ\ 14'$. $C = 90^\circ - 29^\circ\ 14' = 60^\circ\ 46'$. $b = c \div \sin C$. $\therefore$ $\log b = \log c - \log \sin C + 10 = 2.455365$. $\therefore$ $b = 285.341$. $a = c \tan A$. $\therefore$ $\log a = \log c + \log \tan A - 10 = 2.144112$. $\therefore$ $a = 139.35$.

16. $AC = b = 95.75$, $BC = a = 60$. $\sin A = a \div b$. $\therefore$ $\log \sin A = \log a - \log b + 10 = 9.797012$. $\therefore$ $A = 38^\circ\ 48'\ 7''$. $C = 90^\circ - 38^\circ\ 48'\ 7'' = 51^\circ\ 11'\ 53''$. $c = b \cos A$. $\therefore$ $\log c = \log b + \log \cos A - 10 = 1.872853$. $\therefore$ $c = 74.62$.

17. $BC = a = 364.3$, $A = 50^\circ\ 45'$. $b = a \div \sin A$. $\therefore$ $\log b = \log a - \log \sin A + 10 = 2.672498$. $\therefore$ $b = 470.434$.

$c = a \cot \mathrm{A}$. $\therefore \log c = \log a + \log \cot \mathrm{A} - 10 = 2.473699$.
$\therefore c = 297.645$.

GENERAL APPLICATIONS.

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1. The area $= 28 \times 30 \times \sin 75^\circ = 28 \times 30 \times .96593 = 811.37$.

2. The rise $= 463.25 \times \sin 3^\circ = 463.25 \times .05234 = 24.247$.

3. Grade $= \sin^{-1} (150 \div 3000) = \sin^{-1} .05 = 2^\circ\ 52'$.

4. Apothem $= 6 \times \cot \frac{180}{7}^\circ = 6 \times \cot 25^\circ\ 42'\ 51'' = 6 \times 2.07654 = 12.459$. Radius $= 6 \div \sin \frac{180}{7}^\circ = 6 \div \sin 25^\circ\ 42'\ 51'' = 6 \div .43392 = 13.83$.

5. If in one of the 12 triangles which make up the dodecagon, we draw a perpendicular from the end of one of the equal sides upon the other, this perpendicular, being the sine of 30° to the radius 12, will equal $\frac{1}{2}$ of 12, or 6. Hence the area of one of the triangles is $\frac{1}{2} \times 6 \times 12 = 36$, and of the 12, $12 \times 36 = 432$.

6. Height $= 200 \times \tan 47^\circ\ 30' = 200 \times 1.09131 = 218.26$.

7. Angle of elevation $= \tan^{-1} (103 \div 51.5) = \tan^{-1} 2 = 63^\circ\ 26'$.

8. Diameter of base $= 2 \times 126 \times \sin \frac{1}{2} (52^\circ\ 23') = 252 \times \sin 26^\circ\ 11'\ 30'' = 252 \times .44137 = 111.23$. Altitude $= 126 \times \cos 26^\circ\ 11'\ 30'' = 126 \times .89733 = 113.06$.

9. $\mathrm{OM} = \mathrm{EO} \div \sin \mathrm{EMO}$. $\therefore \log \mathrm{OM} = \log \mathrm{EO} - \log \sin \mathrm{EMO} + 10 = 5.377697$. $\therefore \mathrm{OM} = 238{,}615$.

10. Diameter $= 2 \times \mathrm{OM} \times \sin \frac{1}{2} (31'\ 20'') = 2 \times 238{,}615 \times \sin 15'\ 40''$. $\therefore \log$ diam. $= \log 2 + \log 238{,}615 + \log \sin 15'\ 40'' - 10 = 3.337428$. $\therefore$ Diam. $= 5174.8$.

OBLIQUE-ANGLED PLANE TRIANGLES.

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2. $A = 35° 42'$, $B = 76° 27'$, $AB = c = 142$. $c = 180° - (35° 42' + 76° 27') = 67° 51'$. $\sin C : \sin A :: c : a$. $\therefore \log a = a.c.\log \sin C + \log \sin A + \log c - 10 = 1.951655$. $\therefore a = 89.47$. $\sin C : \sin B :: c : b$. $\therefore \log b = a.c.\log \sin C + \log \sin B + \log c - 10 = 2.173323$. $\therefore b = 149.05$.

3. $A = 23° 40' 32''$, $B = 69° 39' 51''$, $c = 100$. $C = 180° - (23° 40' 32'' + 69° 39' 51'') = 86° 39' 37''$. $\sin C : \sin A :: c : a$. $\therefore \log a = a.c.\log \sin C + \log \sin A + \log c - 10 = 1.604487$. $\therefore a = 40.22$. $\sin C : \sin B :: c : b$. $\therefore \log b = a.c.\log \sin C + \log \sin B + \log c - 10 = 1.972789$. $\therefore b = 93.93$.

4. $AB = c = 254.3$, $AC = b = 396.8$, $B = 94° 29'$. $b : c :: \sin B : \sin C$. $\therefore \log \sin C = a.c.\log b + \log c + \log \sin B - 10 = 9.805443$. $\therefore C = 39° 42' 39''$. $A = 180° - (94° 29' + 39° 42' 39'') = 45° 48' 21''$. $\sin B : \sin A :: b : a$. $\therefore \log a = a.c.\log \sin B + \log \sin A + \log b - 10 = 2.455411$. $\therefore a = 285.37$.

5. $B = 40°$, $BD = f = 400$, $DF = b = 350$. $b : f :: \sin B : \sin F$. $\therefore \log \sin F = a.c.\log b + \log f + \log \sin B - 10 = 9.866059$. $\therefore F = 47° 16' 28''$, and $132° 43' 32''$. $D = 180° - (40° + 47° 16' 28'')$, and $180° - (40° + 132° 43' 32'') = 92° 43' 32''$, and $7° 16' 28''$. $\sin B : \sin D_1 :: b : d_1$. $\therefore \log d_1 = a.c.\log \sin B + \log \sin D_1 + \log b - 10 = 2.735510$. $\therefore d_1 = 543.89$. $\sin B : \sin D_2 :: b : d_2$. $\therefore \log d_2 = a.c.\log \sin B + \log \sin D_2 + \log b - 10 = 1.838511$. $\therefore d_2 = 68.95$.

6. $a = 201$, $b = 140$, $B = 36° 44'$. $b : a :: \sin B : \sin A$. $\therefore \log \sin A = a.c.\log b + \log a + \log \sin B - 10 = 9.933836$. $\therefore A = 59° 10' 11''$, and $120° 49' 49''$. $C = 180° - (36° 44' + 59° 10' 11'')$, and $180° - (36° 44' + 120° 49' 49'') = 84° 5' 49''$, and $22° 26' 11''$. $\sin B : \sin C_1 :: b : c_1$.

$\therefore \log c_2 = a.c.\log \sin B + \log \sin C_2 + \log b - 10 = 1.951034$. $\therefore c_2 = 89.34$. $\sin B : \sin C_1 :: b : c_1$. $\therefore \log c_1 = a.c.\log \sin B + \log \sin C_1 + \log b - 10 = 2.367051$. $\therefore c_1 = 232.84$.

7. $a = 180$, $b = 100$, $A = 127° 33'$. $a : b :: \sin A : \sin B$. $\therefore \log \sin B = a.c.\log a + \log b + \log \sin A - 10 = 9.643903$. $\therefore B = 26° 7' 59''$. $C = 180° - (127° 33' + 26° 7' 59'') = 26° 19' 1''$. $\sin A : \sin C :: a : c$. $\therefore \log c = a.c.\log \sin A + \log \sin C + \log a - 10 = 2.002830$. $\therefore c = 100.65$.

8. $a = 30.8$, $b = 54.12$, $B = 36° 42' 11''$. $b : a :: \sin B : \sin A$. $\therefore \log \sin A = a.c.\log b + \log a + \log \sin B - 10 = 9.531653$. $\therefore A = 19° 53' 7''$. $C = 180° - (36° 42' 11'' + 19° 53' 7'') = 123° 24' 42''$. $\sin B : \sin C :: b : c$. $\therefore \log c = a.c.\log \sin B + \log \sin C + \log b - 10 = 1.878447$. $\therefore c = 75.59$. But one triangle, since the side opposite is the greater.

10. $a = 1337.5$, $b = 493$, $A = 69° 46'$. $a : b :: \sin A : \sin B$. $\therefore \log \sin B = a.c.\log a + \log b + \log \sin A - 10 = 9.538891$. $\therefore B = 20° 14' 2''$. $C = 180° - (69° 46' + 20° 14' 2'') = 89° 59' 58''$. $\sin A : \sin C :: a : c$. $\therefore \log c = a.c.\log \sin A + \log \sin C + \log a - 10 = 3.153956$. $\therefore c = 1425.5$.

12. $AB = c = 304$, $BC = a = 280.3$, $B = 100°$. $c + a : c - a :: \tan \frac{1}{2}(C + A) : \tan \frac{1}{2}(C - A)$, or $584.3 : 23.7 :: \tan 40° : \tan \frac{1}{2}(C + A)$. $\therefore \log \tan \frac{1}{2}(C - A) = a.c.\log (c + a) + \log (c - a) + \log \tan \frac{1}{2}(C + A) = 8.531925$. $\therefore \frac{1}{2}(C - A) = 1° 56' 57''$. $C = 40° + 1° 56' 57'' = 41° 56$ $57''$. $A = 40° - 1° 56' 57'' = 38° 3' 3''$. $\sin A : \sin B :: a : b$. $\therefore \log b = a.c.\log \sin A + \log \sin B + \log a - 10 = 2.651139$. $\therefore b = 446.857$.

13. $AB = c = 103$, $AC = b = 126$, $A = 56° 30'$. $b + c : b - c :: \tan \frac{1}{2}(B + C) : \tan \frac{1}{2}(B - C)$, or $229 : 23 :: \tan 61° 45' : \tan \frac{1}{2}(B - C)$. $\therefore \log \tan \frac{1}{2}(B - C) = a.c.\log (b + c) + \log (b - c) + \log \tan \frac{1}{2}(B + C) = 9.271660$. $\therefore \frac{1}{2}(B - C) = 10° 35' 16''$. $B = 61° 45' + 10° 35' 16'' =$

$72° \, 20' \, 16''$, $C = 61° \, 45' - 10° \, 35' \, 16'' = 51° \, 9' \, 44''$. $\sin B : \sin A :: b : a$. $\therefore \log a = a.c.\log \sin B + \log \sin A + \log b - 10 = 2.042448$. $\therefore a = 110.267$.

15. $a = 275$, $b = 300$, $c = 95.6$. $\frac{1}{2}p = 335.3$, $\frac{1}{2}p - a = 60.3$, $\frac{1}{2}p - b = 35.3$, $\frac{1}{2}p - c = 239.7$. $\log k = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - b) + \log(\frac{1}{2}p - c) + a.c.\log \frac{1}{2}p - 10\} = 1.591163$. $\log \tan \frac{1}{2} A = \log k - \log(\frac{1}{2}p - a) + 10 = 9.810846$. $\therefore A = 65° \, 47' \, 55''$. $\log \tan \frac{1}{2} B = \log k - \log(\frac{1}{2}p - b) + 10 = 10.043388$. $\therefore B = 95° \, 42' \, 52''$. $\log \tan \frac{1}{2} C = \log k - \log(\frac{1}{2}p - c) + 10 = 9.211495$. $\therefore C = 18° \, 29' \, 12''$.

16. $BD = a = 500$, $DF = b = 403.7$, $BF = c = 395.75$. $\frac{1}{2}p = 649.725$, $\frac{1}{2}p - a = 149.725$, $\frac{1}{2}p - b = 246.025$, $\frac{1}{2}p - c = 253.975$. $\log k = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - b) + \log(\frac{1}{2}p - c) + a.c.\log \frac{1}{2}p - 10\} = 2.079167$. $\log \tan \frac{1}{2} A = \log k - \log(\frac{1}{2}p - a) + 10 = 9.903873$. $\therefore A = 77° \, 25' \, 14''$. $\log \tan \frac{1}{2} B = \log k - \log(\frac{1}{2}p - b) + 10 = 9.688188$. $\therefore B = 52° \, 0' \, 2''$. $\log \tan \frac{1}{2} C = \log k - \log(\frac{1}{2}p - c) + 10 = 9.674376$. $\therefore C = 50° \, 34' \, 44''$.

OBLIQUE-ANGLED TRIANGLES SOLVED BY MEANS OF RIGHT-ANGLED TRIANGLES.

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1. $BF = d = 123.75$, $DF = b = 500$, $DFB = 120°$. From B let fall a perpendicular BA on DF produced. $BFA = 60°$. $FA = d \cos BFA = 123.75 \times \cos 60° = \frac{1}{2} \times 123.75 = 61.875$. $BA = d \sin BFA = 123.75 \times \sin 60° = \frac{1}{2}\sqrt{3} \times 123.75 = 107.17$. $FBA = 90° - 60° = 30°$. $DA = b + FA = 561.875$. $\tan D = BA \div DA$. $\therefore \log \tan D = \log BA - \log DA + 10 = 9.280453$. $\therefore D = 10° \, 47' \, 57''$. $B = 60° - 10° \, 47' \, 57'' = 49° \, 12' \, 3''$. $BD = BA \div \sin D$. $\therefore \log BD = \log BA - \log \sin D + 10 = 2.757400$. $\therefore BD = 572.005$.

2. $A = 70° \, 21'$, $B = 54° \, 22'$, $BC = a = 125$. From B let fall a perpendicular BD upon AC. $C = 180° - (70° \, 21' +$

$54° 22')=55° 17'$. $BD=a \sin C$. $\therefore \log BD=\log a+\log \sin C-10=2.011770$. $CD=a \cos C$. $\therefore \log CD=\log a+\log \cos C-10=1.852418$. $\therefore CD=71.19$. $c=BD \div \sin A$. $\therefore \log c=\log BD-\log \sin A+10=2.037828$. $\therefore c=109.1$. $AD=BD \cot A$. $\therefore \log AD=\log BD+\log \cot A-10=1.564520$. $\therefore AD=36.69$. $AC=b=CD+AD=71.19+36.69=107.88$.

3. $AB=c=98$, $BC=a=95.12$, $C=33° 21'$. From B let fall a perpendicular BD upon AC. $CBD=90°-C=90°-33° 21'=56° 39'$. $BD=a \sin C$. $\therefore \log BD=\log a+\log \sin C-10=1.718439$. $CD=a \cos C$. $\therefore \log CD=\log a+\log \cos C-10=1.900129$. $\therefore CD=79.46$. $\sin A=BD \div c$. $\log \sin A=\log BD-\log c+10=9.727213$. $\therefore A=32° 14' 55''$. $DBA=90°-32° 14' 55''=57° 45' 5''$. $B=DBA+CBD=57° 45' 5''+56° 39'=114° 24' 5''$. $DA=BD \cot A$. $\therefore \log DA=\log BD+\log \cot A-10=1.918465$. $\therefore DA=82.88$. $b=CD+DA=79.46+82.88=162.34$.

4. $DP=c \cos D$. $\therefore \log DP=\log c+\log \cos D-10=2.599817$. $\therefore DP=397.94$. $CP=d \cos C'$. $\therefore \log CP=\log d+\log \cos C'-10=2.355213$. $\therefore CP=226.58$. $CD=397.94-226.58=171.36$. $C'D=397.94+226.58=624.5$.

5. $AB=c=460$, $BC=a=340$, $AC=b=280$. From C let fall a perpendicular CD upon AB. Call $AD=n$, $DB=m$. Then $c:a+b::a-b:m-n$, or $460:620::60:m-n$. $\therefore m-n=80.87$. $m=270.435$, $n=189.565$. $\cos A=n \div b$. $\therefore \log \cos A=\log n-\log b+10=9.830600$. $\therefore A=47° 23' 20''$. $\cos B=m \div a$. $\therefore \log \cos B=\log m-\log a+10=9.900584$. $\therefore B=37° 18' 26''$. $C=180°-(47° 23' 20''+37° 18' 26'')=95° 18' 14''$.

6. $c=40$, $a=34$, $b=25$. $c:a+b::a-b:m-n$, or $40:59::9:m-n$. $\therefore m-n=13.275$. $m=26.6375$, $n=13.3625$. $\cos A=n \div b$. $\therefore \log \cos A=\log n-\log b+10=9.727948$. $\therefore A=57° 41' 24''$. $\cos B=m \div a$.

$\therefore \log \cos \mathrm{B} = \log m - \log a + 10 = 9.894014$. $\therefore \mathrm{B} = 38° 25' 20''$. $\mathrm{C} = 180° - (57° 41' 24'' + 38° 25' 20'') = 83° 53' 16''$.

7. $c = 390$, $a = 350$, $b = 270$. $c : a + b :: a - b : m - n$, or $390 : 620 :: 80 : m - n$. $\therefore m - n = 127.18$. $m = 258.59$, $n = 131.41$. $\cos \mathrm{B} = m \div a$. $\therefore \log \cos \mathrm{B} = \log m - \log a + 10 = 9.868544$. $\therefore \mathrm{B} = 42° 22' 6''$. $\cos \mathrm{A} = n \div b$. $\therefore \log \cos \mathrm{A} = \log n - \log b + 10 = 9.687264$. $\therefore \mathrm{A} = 60° 52' 33''$. $\mathrm{C} = 180° - (42° 22' 6'' + 60° 52' 33'') = 76° 45' 21''$.

8. $a = 540$, $b = 450$, $\mathrm{C} = 80°$. From A let fall a perpendicular AD upon CB. $\mathrm{CD} = b \cos \mathrm{C}$. $\therefore \log \mathrm{CD} = \log b + \log \cos \mathrm{C} - 10 = 1.892883$. $\therefore \mathrm{CD} = 78.14$. $\mathrm{AD} = b \sin \mathrm{C}$. $\therefore \log \mathrm{AD} = \log b + \log \sin \mathrm{C} - 10 = 2.646564$. $\mathrm{DB} = a - \mathrm{CD} = 540 - 78.14 = 461.86$. $\tan \mathrm{B} = \mathrm{AD} \div \mathrm{DB}$. $\therefore \log \tan \mathrm{B} = \log \mathrm{AD} - \log \mathrm{DB} + 10 = 9.982054$. $\therefore \mathrm{B} = 43° 49'$. $\mathrm{A} = 100° - 43° 49' = 56° 11'$. $c = \mathrm{AD} \div \sin \mathrm{B}$. $\therefore \log c = \log \mathrm{AD} - \log \sin \mathrm{B} + 10 = 2.806236$. $\therefore c = 640.08$.

9. $a = 109$, $b = 76$, $\mathrm{C} = 101° 30'$. From A let fall a perpendicular AD upon BC produced. Then $\mathrm{AD} = b \sin \mathrm{ACD}$. $\therefore \log \mathrm{AD} = \log b + \log \sin \mathrm{ACD} - 10 = 1.872007$. $\mathrm{CD} = b \cos \mathrm{ACD}$. $\therefore \log \mathrm{CD} = \log b + \log \cos \mathrm{ACD} - 10 = 1.180469$. $\therefore \mathrm{CD} = 15.15$. $\mathrm{BD} = a + \mathrm{CD} = 109 + 15.15 = 124.15$. $\tan \mathrm{B} = \mathrm{AD} \div \mathrm{BD}$. $\therefore \log \tan \mathrm{B} = \log \mathrm{AD} - \log \mathrm{BD} + 10 = 9.778060$. $\therefore \mathrm{B} = 30° 57' 30$. $\mathrm{A} = 180° - (101° 30' + 30° 57' 30'') = 47° 32' 30''$. $c = \mathrm{AD} \div \sin \mathrm{B}$. $\therefore \log c = \log \mathrm{AD} - \log \sin \mathrm{B} + 10 = 2.160694$. $\therefore c = 144.8$.

FUNCTIONS OF THE ANGLES OF A TRIANGLE IN TERMS OF ITS SIDES.

Page 60, 94.

2. $a = 6$, $b = 5$, $c = 4$. $\frac{1}{2}p = 7.5$, $\frac{1}{2}p - a = 1.5$, $\frac{1}{2}p - b = 2{,}5$, $\frac{1}{2}p - c = 3.5$. $\sin\frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{bc}}$. ∴ $\log\sin\frac{1}{2}A = \frac{1}{2}\{\log(\frac{1}{2}p - b) + \log(\frac{1}{2}p - c) + a.c.\log b + a.c.\log c\} = 9.820489$. ∴ $A = 82° 49' 10''$. $\sin\frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - c)}{ac}}$. ∴ $\log\sin\frac{1}{2}B = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - c) + a.c.\log a + a.c.\log c\} = 9.669974$. ∴ B $55° 46' 16''$. $\sin\frac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)}{ab}}$. ∴ $\log\sin\frac{1}{2}C = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - b) + a.c.\log a + a.c.\log b\} = 9.548455$. ∴ $C = 41° 24' 34''$.

3. $a = 8601.5$, $b = 7068$, $c = 4082$. $\frac{1}{2}p = 9875.75$, $\frac{1}{2}p - a = 1274.25$, $\frac{1}{2}p - b = 2807.75$, $\frac{1}{2}p - c = 5793.75$. $\sin\frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{bc}}$. ∴ $\log\sin\frac{1}{2}A = \frac{1}{2}\{\log(\frac{1}{2}p - b) + \log(\frac{1}{2}p - c) + a.c.\log b + a.c.\log c\} = 9.875574$. ∴ $A = 97° 20' 4''$. $\sin\frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - c)}{ac}}$. ∴ $\log\sin\frac{1}{2}B = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - c) + a.c.\log a + a.c.\log c\} = 9.661384$. ∴ $B = 54° 35' 12''$. $\sin\frac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p - a)(\frac{1}{2}p - b)}{ab}}$. ∴ $\log\sin\frac{1}{2}C = \frac{1}{2}\{\log(\frac{1}{2}p - a) + \log(\frac{1}{2}p - b) + a.c.\log a + a.c.\log b\} = 9.384871$. ∴ $C = 28° 4' 44''$.

4. $a = .5123864$, $b = .3538971$, $c = .3090507$. $\frac{1}{2}p = .5876671$, $\frac{1}{2}p - a = .0752807$, $\frac{1}{2}p - b = .23377$, $\frac{1}{2}p - c = .2786164$. $\tan\frac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p - b)(\frac{1}{2}p - c)}{\frac{1}{2}p(\frac{1}{2}p - a)}}$. ∴ $\log\tan\frac{1}{2}A = \frac{1}{2}\{\log(\frac{1}{2}p - b) + \log(\frac{1}{2}p - c) + a.c.\log\frac{1}{2}p + a.c.$

$\log\left(\frac{1}{2}p-a\right)\} = 10.083988$. $\therefore$ $A = 101^\circ\,0'\,44''$. $\tan\frac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p-a)\,(\frac{1}{2}p-c)}{\frac{1}{2}p\,(\frac{1}{2}p-b)}}$. $\therefore$ $\log\tan\frac{1}{2}B = \frac{1}{2}\{\log(\frac{1}{2}p-a) + \log\frac{1}{2}\ p-c) + a.c.\log\frac{1}{2}p + a.c.\log(\frac{1}{2}p-b)\} = 9.591883$. $\therefore$ $B = 42^\circ\,41'\,6''$. $\tan\frac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p-a)\,(\frac{1}{2}p-b)}{\frac{1}{2}p\,(\frac{1}{2}p-c)}}$. $\therefore$ $\log\tan\frac{1}{2}C = \frac{1}{2}\{\log(\frac{1}{2}p-a) + \log(\frac{1}{2}p-b) + a.c.\log\frac{1}{2}p + a.c.\log(\frac{1}{2}p-c)\} = 9.515670$. $\therefore$ $C = 36^\circ\,18'\,10''$.

AREA OF PLANE TRIANGLES.

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1. $\log \text{area} = \log 125.81 + \log 57.65 + \log\sin 57^\circ\,25' + a.c.\log 2 - 20 = 3.485110$. $\therefore$ area = 3055.7.

2. $\log \text{area} = \log 103.5 + \log 90 + \log\sin 100^\circ + a.c.\log 2 - 20 = 3.661504$. $\therefore$ area = 4586.74.

3. Area $= \sqrt{60 \times 30 \times 20 \times 10} = 600$ sq. ft. $= 66\frac{2}{3}$ sq. yds.

4. Area $= \sqrt{45 \times 25 \times 15 \times 5} = 75\sqrt{15} = 290.4737$.

5. Area $= \frac{1}{2} \times 30 \times 40 \times \sin 28^\circ\,57' = 600 \times .48405 = 290.43$.

6. $\log \text{area} = \log 25 + \log 21.25 + \log\sin 45^\circ + a.c.\log 2 - 20 = 2.273754$. $\therefore$ area = 187.83 sq. ft. = 20.87 sq. yds.

7. $a = 32$, $B = 80^\circ$, $C = 60^\circ$. $A = 180^\circ - (80^\circ + 60^\circ) = 40.^\circ$ $\text{area} = \frac{1}{2}\,a\cdot b\cdot\sin C = \frac{1}{2}\,a\cdot\frac{a\sin B}{\sin A}\cdot\sin C$. $\therefore$ $\log \text{area} = 2\log a + \log\sin B + \log\sin C + a.c.\log\sin A + a.c.\log 2 - 30 = 2.832085$. $\therefore$ area = 679.34.

8. $a = 45$, $B = 70^\circ$, $C = 69^\circ\,40'$. $A = 180^\circ - (70^\circ + 69^\circ\,40') = 40^\circ\,20'$. $\log \text{area} = 2\log a + \log\sin B + \log\sin C + a.c.\log\sin A + a.c.\log 2 - 30 = 3.139378$. $\therefore$ area = 1378.41.

9. $\text{A} = 73^\circ\ 10'$, $\text{B} = 90^\circ\ 50'$, $b = 75.3$. $\text{C} = 180^\circ - (73^\circ\ 10' + 90^\circ\ 50') = 16^\circ$. $\log \text{area} = 2 \log b + \log \sin \text{A} + \log \sin \text{C} + a.\ c.\ \log \sin \text{B} + a.\ c.\ \log 2 - 30 = 2.873925$. $\therefore \text{area} = 748.04$.

PRACTICAL APPLICATIONS.

Page 62.

1. Length of arc $1^\circ = \text{CL} \cos \text{ECL} \times 2\pi \div 360$. $\therefore \log \text{arc } 1^\circ = \log \text{CL} + \log \cos \text{ECL} + \log \pi + a.\ c.\ \log 180 - 20 = 1.708395$. $\therefore \text{arc } 1^\circ = 51.1$.

3. Let $\text{CNP} = \alpha$, $\text{CN'P} = \beta$, $\text{CPN} = x$, $\text{CPN'} = y$. Then $\sin x : \sin \alpha :: \text{CN} : \text{CP}$; also $\sin y : \sin \beta :: \text{CN} : \text{CP}$. $\therefore \sin x : \sin \alpha :: \sin y : \sin \beta$, or $\sin x : \sin y :: \sin \alpha : \sin \beta$. By composition and division, $\sin x + \sin y : \sin x - \sin y :: \sin \alpha + \sin \beta : \sin \alpha - \sin \beta$, or $\tan \frac{1}{2}(x + y) : \tan \frac{1}{2}(x - y) :: \tan \frac{1}{2}(\alpha + \beta) : \tan \frac{1}{2}(\alpha - \beta)$. $x + y = \text{PNZ} + \text{PN'Z'} - \text{NCN'} = 1^\circ\ 23'\ 18''$. $\frac{1}{2}(x + y) = 41'\ 39''$. $\alpha - \beta = \text{PN'Z'} - \text{PNZ} = 3^\circ\ 48'\ 36''$. $\frac{1}{2}(\alpha - \beta) = 1^\circ\ 54'\ 18''$. $\alpha + \beta = 360^\circ - (\text{PNZ} + \text{PN'Z'}) = 266^\circ\ 22'\ 42''$. $\frac{1}{2}(\alpha + \beta) = 133^\circ\ 11'\ 21''$. Now, $\log \tan \frac{1}{2}(x - y) = \log \tan \frac{1}{2}(x + y) + a.\ c.\ \log \tan \frac{1}{2}(\alpha + \beta) + \log \tan \frac{1}{2}(\alpha - \beta) - 10 = 6.577824$. $\therefore \frac{1}{2}(x - y) = 1'\ 18''.028$. $\tan \frac{1}{2}(x - y)$ being negative, $y > x$. $\therefore y = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) = 42'\ 57''.028$; $x = \frac{1}{2}(x + y) - \frac{1}{2}(x - y) = 40'\ 20''.972$. Now we may take $\sin x : \sin \alpha :: \text{CN} : \text{CP}$, whence $\log \text{CP} = a.c.\log \sin x + \log \sin \alpha + \log \text{CN} - 10$, or $\sin y : \sin \beta :: \text{CN} : \text{CP}$, whence $\log \text{CP} = a.\ c.\ \log \sin y + \log \sin \beta + \log \text{CN} - 10$. $\therefore \log \text{CP} = 5.376493$, and $\text{CP} = 237{,}954$.

4. $\text{BD} = \dfrac{\text{AB} \sin \text{DAB}}{\sin \text{ADB}}$; $\text{CD} = \text{BD} \sin \text{DBC}$. $\therefore \text{CD} = \dfrac{\text{AB} \sin \text{DAB} \sin \text{DBC}}{\sin \text{ADB}}$. $\therefore \log \text{CD} = \log \text{AB} + \log \sin \text{DAB} + \log \sin \text{DBC} + a.\ c.\ \log \sin \text{ADB} - 20 = 2.769093$. $\therefore \text{CD} = 587.61$.

5. The area of any regular polygon of n sides, the length of each side being a, is easily seen to be $\frac{na^2}{4} \cot \frac{180^\circ}{n}$. In the hexagon, area $= \frac{6 \times 10^2}{4} \cot 30^\circ = 150 \times 1.73205 = 259.8$. In the octagon, area $= \frac{8 \times 10^2}{4} \cot 22^\circ\ 30' = 200 \times 2.41421 = 482.84$.

6. In the pentagon, area $= \frac{5 \times 12^2}{4} \cot 36^\circ = 180 \times 1.37638 = 247.74$. In the decagon, area $= \frac{10 \times 12^2}{4} \cot 18^\circ = 360 \times 3.07768 = 1107.96$.

7. $ACB = 180^\circ - (CAB + CBA) = 45^\circ$. $\sin ACB : \sin ABC :: AB : AC$. $\therefore \log AC = a.\ c.\ \log \sin ACB + \log ABC + \log AB - 10 = 1.929978$. $\therefore AC = 85.11$. $ADB = 180^\circ - (DAB + ABD) = 30^\circ$. $\sin ADB : \sin ABD :: AB : AD$. $\therefore \log AD = a.\ c.\ \log \sin ADB + \log \sin ABD + \log AB - 10 = 2.246965$. $\therefore AD = 176.59$. $AD + AC : AD - AC :: \tan \frac{1}{2} (ACD + ADC) : \tan \frac{1}{2} (ACD - ADC)$, or $261.7 : 91.48 :: \tan 57^\circ : \tan \frac{1}{2} (ACD - ADC)$. $\therefore \log \tan \frac{1}{2} (ACD - ADC) = a.\ c.\ \log (AD + AC) + \log (AD - AC) + \log \tan \frac{1}{2} (ACD + ADC) - 10 = 9.731005$. $\therefore \frac{1}{2} (ACD - ADC) = 28^\circ\ 17'\ 33''$. $ACD = 85^\circ\ 17'\ 33''$, $ADC = 28^\circ\ 42'\ 27''$. $\sin ADC : \sin CAD :: AC : CD$. $\therefore \log CD = a.\ c.\ \log \sin ADC + \log \sin CAD + \log AC - 10 = 2.209161$. $\therefore CD = 161.868$.

8. In triangle ADB, $ADB = 180^\circ - (DAB + ABD) = 123^\circ\ 45'$. $\sin ADB : \sin ABD :: AB : AD$. $\therefore \log AD = a\,.\,c\,.\ \log \sin ADB + \log \sin ABD + \log AB - 10 = 2.727983$. $\therefore AD = 534.543$. $\cos BAC = \frac{\overline{AB}^2 + \overline{AC}^2 - \overline{BC}^2}{2\, AB \times AC} = \frac{7}{8}$. $\therefore BAC = 28^\circ\ 57'\ 18''$. $DAC = BAC - BAD = 6^\circ\ 27'\ 18''$. $AC + AD : AC - AD :: \tan \frac{1}{2} (ADC + ACD) : \tan \frac{1}{2} (ADC - ACD)$, or $1134.543 : 65.457 :: \tan 86^\circ\ 46'\ 21'' : \tan \frac{1}{2} (ADC -$

ACD). $\therefore \log \tan \frac{1}{2}(\text{ADC}-\text{ACD}) = a.\,c. \log(\text{AC}+\text{AD}) + \log(\text{AC}-\text{AD}) + \log\tan\frac{1}{2}(\text{ADC}+\text{ACD}) - 10 = 10.009930$. $\therefore \frac{1}{2}(\text{ADC}-\text{ACD}) = 45^\circ\ 39'\ 18''$. $\text{ACD} = 41^\circ\ 7'\ 3''$. $\sin \text{APC} : \sin \text{ACP} :: \text{AC} : \text{AP}$. $\therefore \log \text{AP} = a.\,c. \log \sin \text{APC} + \log \sin \text{ACP} + \log \text{AC} - 10 = 2.851377$. $\therefore \text{AP} = 710.19$. $\sin \text{APC} : \sin \text{CAP} :: \text{AC} : \text{CP}$. $\therefore \log \text{CP} = a.\,c. \log \sin \text{APC} + \log \sin \text{CAP} + \log \text{AC} - 10 = 3.018086$. $\therefore \text{CP} = 1042.52$. $\text{BAP} = \text{CAP} - \text{CAB} = 76^\circ\ 10'\ 39''$. $\text{APB} = \text{APC} + \text{CPB} = 56^\circ\ 15'$. $\sin \text{APB} : \sin \text{BAP} :: \text{AB} : \text{BP}$. $\therefore \log \text{BP} = a.\,c. \log \sin \text{APB} + \log \sin \text{BAP} + \log \text{AB} - 10 = 2.970481$. $\therefore \text{BP} = 934.29$.

Another form of solution may be outlined as follows: Find the angle ACB. Let $\text{BC} = b$, $\text{CB} = a$, $\text{APC} = \alpha$, $\text{CPB} = \beta$, $\text{CAP} = x$, $\text{CBP} = y$; then $x + y = 360^\circ - (\alpha + \beta + \text{C})$. Now $\sin \alpha : \sin x :: b : \text{PC}$, also $\sin \beta : \sin y :: a : \text{PC}$. $\therefore \text{PC} = \frac{b \sin x}{\sin \alpha} = \frac{a \sin y}{\sin \beta}$. From this, $\frac{\sin x}{\sin y} = \frac{a \sin \alpha}{b \sin \beta} = \tan \varphi$ (say). By composition and division,

$$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \frac{1}{2}(x - y)}{\tan \frac{1}{2}(x + y)} = \frac{\tan \varphi - 1}{\tan \varphi + 1} = \tan(\varphi - 45^\circ).$$

$\therefore x - y$ can be found, and consequently x and y. Then, $\sin \alpha : \sin x :: b : \text{PC}$, $\sin \alpha : \sin \text{ACP} :: b : \text{AP}$, $\sin \beta : \sin \text{PCB} :: a : \text{PB}$.

9. By forming a right-angled triangle, with the line from the top of the mountain to the centre of the earth as the hypotenuse, the radius and the tangent as the other sides, we see that the angle at the centre is equal to the angle of depression, O. Let x be the radius; then $\cos \text{O} = \frac{x}{x + 3}$. $\therefore x = \frac{3 \cos \text{O}}{1 - \cos \text{O}} = 3 \cot \text{O} \cot \frac{1}{2} \text{O}$, or diam. $= 6 \cot \text{O} \cot \frac{1}{2} \text{O}$. $\therefore$ log diam. $= \log 6 + \log \cot \text{O} + \log \cot \frac{1}{2} \text{O} - 20 = 3.900818$. $\therefore$ Diam. $= 7958.3$.

10. Diameter $= 2 \times 91{,}430{,}000 \times \sin 16'\ 1''.7$. $\therefore$ Diam. $= \log^{-1} 5.930732 = 852{,}574$.

11. Assuming radius $= 3962.8$, we have to solve a triangle whose hypotenuse, $h = 3962.8 + \frac{486}{5280} = 3962.8904$, and side, $r = 3962.8$. $\therefore$ other side $= \sqrt{h^2 - r^2} = \sqrt{(h+r)(h-r)}$. $\therefore$ log other side $= \frac{1}{2}\{\log(h+r) + \log(h-r)\} = 1.427602$. $\therefore$ distance $= 27.-$ miles.

12. Referring to FIG. in Ex. 4, $\text{CD} = \frac{\text{AB} \sin \text{DAB} \sin \text{DBC}}{\sin \text{ADB}}$. $\therefore \log \text{CD} = \log \text{AB} + \log \sin \text{DAB} + \log \sin \text{DBC} + a.c. \log \sin \text{ADB} - 20 = 2.971163$. $\therefore \text{CD} = 935.757$.

13. cos angle opposite $100 = \frac{60^2 + 80^2 - 100^2}{2 \cdot 60 \cdot 80} = 0$. $\therefore$ angle $= 90°$, and the figure is a rectangle.

14. Height $= 8 \times \tan 10° = 8 \times .176327 = 1.41$.

15. Distance $= 3962.8 \div \sin 8''.94 = \log^{-1}\{\log 3962.8 + a.c. \log \sin 8''.94\} = \log^{-1} 7.961089 = 91{,}430{,}000$.

SPHERICAL TRIGONOMETRY.

PROJECTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Page 67, 100.

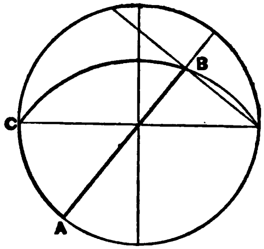

Fig. 14 *a*.

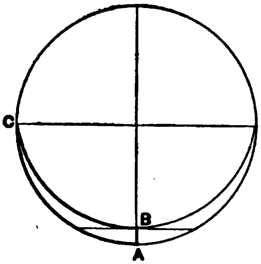

Fig. 15 *a*.

QUERIES. Yes. dd' and AA' will always intersect.

2, 3, and 4. See FIGS. 14a, 15a, and 16a.

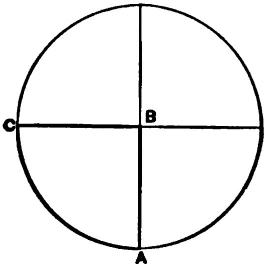

Fig. 16 *a*.

Page 68, 101.

QUERIES. No, to all.

1, 2, 4, and 5. All impossible, because dd' and AA' do not intersect. Query. 90°, because dd' and AA' must then coincide in order to have a triangle.

Page 69, 103.

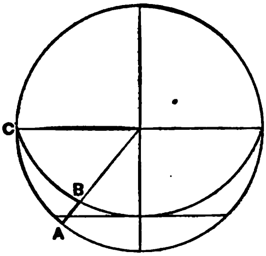

Fig. 17 *a*.

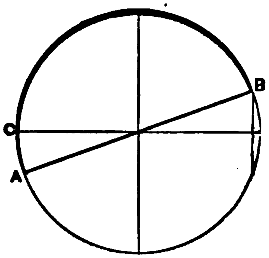

Fig. 18 *a*.

Query. Yes.

2, 3, and 4. See Figs. 17*a*, 18*a*, and 19*a*.

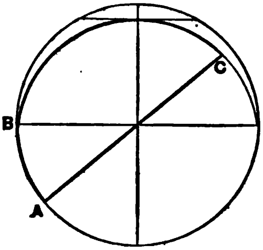

Fig. 19 *a*.

Page 70 104.

Queries. When the small circumference and the ellipse intersect. When they are tangent to each other. When they do not meet. Right-angled. $B > 90°$, $b > B$. $B < 90°$, $b < B$. No. No.

5. Impossible.

PROJECTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

Page 71, 106.

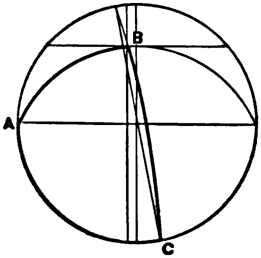

Fig. 20 a.

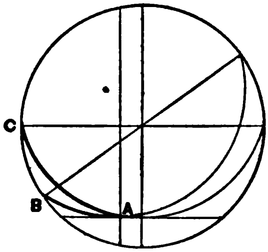

Fig. 21 a.

Query. Yes.

1 and 2. See Figs. 20 *a* and 21 *a*.

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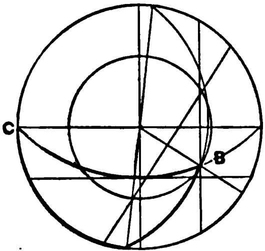

Fig. 22 a.

2, 4, and 5. Impossible.

3. See Fig. 22 *a*.

Page 73, 109.

Queries. In both cases, dd' and ee' do not intersect.

1. Fig. 40 illustrates the method, projecting upon *a*.

3 and 4. Impossible.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Page 75, 117.

2. The circular parts are comp C, comp *a*, and *b*. comp C is the middle part, and the extremes are adjacent.

3. The circular parts are *c*, *b*, and comp B. *c* is the middle part, and the extremes are adjacent.

4. Answers same as in 2.

5. comp B, comp C, and b, are the circular parts. comp B is the middle part, and the extremes are opposite.

6. comp a and comp B. c and comp C. comp a and comp C; b and comp B. b and comp C; c and comp a. c and comp B; b and comp a. b and c; comp B and comp C.

8. n, o, comp m, comp N, and comp O. o, comp O, and comp m are the circular parts, and the extremes are opposite.

9. o and comp N. n and comp m. n and o; comp O and comp N. comp m and comp O; n and comp N. comp m and comp N; o and comp O.

Page 82, 128.

2. $a = 86° \, 51'$, $B = 18° \, 3' \, 32''$. $\sin b = \sin a \sin B$. $b < 90°$. $\log \sin b = \log \sin a + \log \sin B - 10 = 9.490309$. $\therefore b = 18° \, 1' \, 50''$. $\cot C = \cos a \div \cot B$. $C < 90°$. $\log \cot C = \log \cos a - \log \cot B + 10 = 8.252833$. $\therefore C = 88° \, 58' \, 28''$. $\tan c = \cos B \div \cot a$. $c < 90°$. $\log \tan c = \log \cos B - \log \cot a + 10 = 11.237476$. $\therefore c = 86° \, 41' \, 15''$. *Check.* $\sin b = \tan c \cot C$, or $\log \sin b = \log \tan c + \log \cot C - 10$.

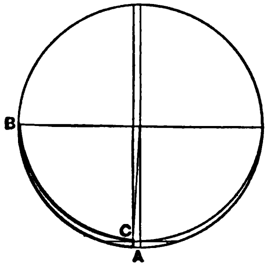

Fig. 23 *a*.

3. $b = 155° \, 27' \, 54''$, $c = 29° \, 46' \, 8''$. $\cos a = \cos b \cos c$. $a > 90°$. $\log \cos a = \log \cos b + \log \cos c - 10 = 9.897439$. $\therefore a = 142° \, 9' \, 13''$. $\cot C = \sin b \div \tan c$. $C < 90°$. $\log \cot C = \log \sin b - \log \tan c + 10 = 9.860925$. $\therefore C = 54° \, 1' \, 16''$. $\cot B = \sin c \div \tan b$. $B > 90°$. $\log \cot B = \log \sin c - \log \tan b + 10 = 10.036515$. $\therefore B = 137° \, 24' \, 21''$. *Check.* $\cos a = \cot B \cot C$, or $\log \cos a = \log \cot B + \log \cot C - 10$.

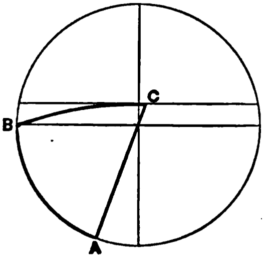

Fig. 24 *a*.

4. $c = 73^\circ\ 41'\ 35''$, $B = 99^\circ\ 17'\ 33''$. $\cot a = \cos B \div \tan c$. $a > 90^\circ$. $\log \cot a = \log \cos B - \log \tan c + 10 = 8.674307$. $\therefore a = 92^\circ\ 42'\ 17''$. $\cos C = \cos c \sin B$. $C < 90^\circ$. $\log \cos C = \log \cos c + \log \sin B - 10 = 9.442634$. $\therefore C = 73^\circ\ 54'\ 46''$. $\tan b = \sin c \div \cot B$. $b > 90^\circ$. $\log \tan b = \log \sin c - \log \cot B + 10 = 10.768327$. $\therefore b = 99^\circ\ 40'\ 30''$. *Check.* $\cos C = \tan b \cot a$, or $\log \cos C = \log \tan b + \log \cot a - 10$.

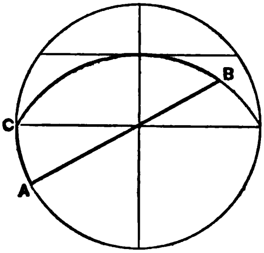

Fig. 25 *a*.

5. $B = 47^\circ\ 13'\ 43''$, $C = 126^\circ\ 40'\ 24''$. $\cos a = \cot B \cot C$. $a > 90^\circ$. $\log \cos a = \log \cot B + \log \cot C - 10 = 9.838135$. $\therefore a = 133^\circ\ 32'\ 26''$. $\cos c = \cos C \div \sin B$. $c > 90^\circ$. $\log \cos c = \log \cos C - \log \sin B + 10 = 9.910421$. $\therefore c = 144^\circ\ 27'\ 4''$. $\cos b = \cos B \div \sin C$. $b < 90^\circ$. $\log \cos b = \log \cos B - \log \sin C + 10 = 9.927714$. $\therefore b = 32^\circ\ 8'\ 55''$. *Check.* $\cos a = \cos b \cos c$, or $\log \cos a = \log \cos b + \log \cos c - 10$.

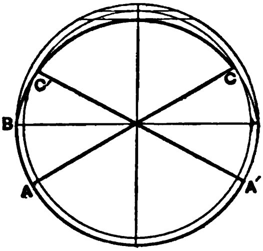

Fig. 26 *a*.

7. $b = 160^\circ$, $B = 150^\circ$. Two solutions. $\sin a = \sin b \div \sin B$. $a < 90^\circ, > 90^\circ$. $\log \sin a = \log \sin b - \log \sin B + 10 = 9.835082$. $\therefore a = 43^\circ\ 9'\ 37'', 136^\circ\ 50'\ 23''$. $\sin c = \tan b \cot B$. $c > 90^\circ, < 90^\circ$. $\log \sin c = \log \tan b + \log \cot B - 10 = 9.799627$. $\therefore c = 140^\circ\ 55'\ 9'', 39^\circ\ 4'\ 51''$. $\sin C = \cos B \div \cos b$. $C > 90^\circ, < 90^\circ$. $\log \sin C = \log \cos B - \log \cos b + 10 = 9.964545$. $\therefore C = 112^\circ\ 50'\ 17'', 67^\circ\ 9'\ 43''$.

Check. $\sin c = \sin a \sin C$, or $\log \sin c = \log \sin a + \log \sin C - 10$.

8 and 9. No solution, since $\sin b > \sin B$.

10. $b = B = 65^\circ\ 23'\ 12''$. Projection like Fig. 35, p. 109.

$\sin a = \sin b \div \sin B = 1$. $\therefore\ a = 90°$. $\sin c = \tan b \cot B = 1$. $\therefore\ c = 90°$. $\sin C = \cos B \div \cos b = 1$. $\therefore\ C = 90°$.

11. $C = 60°\ 47'\ 24''.3$, $B = 57°\ 16'\ 20''.2$. $\cos a = \cot B \cot C$. $a < 90°$. $\log \cos a = \log \cot B + \log \cot C - 10 = 9.555485$. $\therefore\ a = 68°\ 56'\ 28''.9$. $\cos b = \cos B \div \sin C$. $b < 90°$. $\log \cos b = \log \cos B - \log \sin C + 10 = 9.791980$. $\therefore\ b = 51°\ 43'\ 36''.3$. $\cos c = \cos C \div \sin B$. $c < 90°$. $\log \cos c = \log \cos C - \log \sin B + 10 = 9.763505$. $\therefore\ c = 54°\ 32'\ 32''.2$.

Check. $\cos a = \cos b \cos c$, or $\log \cos a = \log \cos b + \log \cos c - 10$.

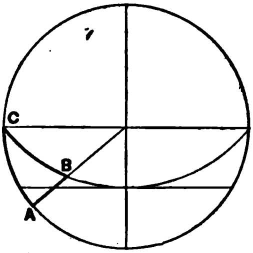

Fig. 27 *a*.

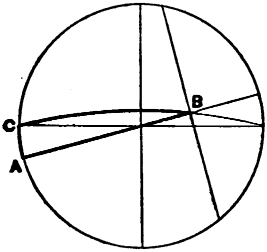

Fig. 28 *a*.

12. $b = 16°$, $c = 116°$. $\cos a = \cos b \cos c$. $a > 90°$. $\log \cos a = \log \cos b + \log \cos c - 10 = 9.624684$. $\therefore\ a = 114°\ 55'\ 25''$. $\cot B = \sin C \div \tan b$. $B < 90°$. $\log \cot B = \log \sin c - \log \tan b + 10 = 10.496164$. $\therefore\ B = 17°\ 41'\ 40''$. $\cot C = \sin b \div \tan c$. $C > 90°$. $\log \cot C = \log \sin b - \log \tan c + 10 = 9.128520$. $\therefore\ C = 97°\ 39'\ 24''$. *Check.* $\cos a = \cot B \cot C$, or $\log \cos a = \log \cot B + \log \cot C - 10$.

QUADRANTAL TRIANGLES.

Page 84, 129.

1. $\quad = 104°\ 18'$, $C' = 161°\ 23'$. $\cot a' = \cos C' \div \tan b'$. $a' < 90°$. $\therefore\ \log \cot a' = \log \cos C' - \log \tan b' + 10 = 9.383024$. $\therefore\ a' = 76°\ 25'\ 11''$. $\cos B' = \cos b' \sin C'$. B'

$> 90°$. $\log \cos B' = \log \cos b' + \log \sin C' - 10 = 8.896805$. $\therefore B' = 94° 31' 21''$. $\tan c' = \sin b' \div \cot C'$. $c' > 90°$. $\log \tan c' = \log \sin b' - \log \cot C' + 10 = 9.513782$. $\therefore c' = 161° 55' 20''$. *Check.* $\cos B' = \tan c' \cot C$, or $\log \cos B' = \log \tan c' + \log \cot C' - 10$.

2. $a = 90°$, $C = 42° 10'$, $A = 115° 20'$. $A' = 180° - a = 90°$, $c' = 180° - C = 137° 50'$, $a' = 180° - A = 64° 40'$. $\cos b' = \cos a' \div \cos c'$. $b' > 90°$. $\log \cos b' = \log \cos a' - \log \cos c' + 10 = 9.761393$. $\therefore b' = 125° 15' 36''$. $\sin C' = \sin c' \div \sin a'$. $C' > 90°$. $\log \sin C' = \log \sin c' - \log \sin a' + 10 = 9.870821$. $\therefore C' = 132° 2' 13''$. $\cos B' = \tan c' \cot a'$. $B' > 90°$. $\log \cos B' = \log \tan c' + \log \cot a' - 10 = 9.632214$. $\therefore B' = 115° 23' 20''$.

Check. $\cos B' = \cos b' \sin C'$, or $\log \cos B' = \log \cos b' + \log \sin C' - 10$. $B = 180° - b' = 54° 44' 24''$, $b = 180° - B' = 64° 36' 40''$, $c = 180° - C' = 47° 57' 47''$.

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES BY NAPIER'S RULES FOR RIGHT-ANGLED SPHERICAL TRIANGLES.

Page 94, 142.

4. $A = 128° 45'$, $C = 30° 35'$, $a = 68° 50'$. $a' = 180° - A = 51° 15'$, $c' = 180° - C = 149° 25'$, $A' = 180° - a = 111° 10'$. $\sin p = \sin c' \sin A'$. $p > 90°$. $\log \sin p = \log \sin c' + \log \sin A' - 10 = 9.676204$. $\therefore p = 151° 40' 32''$. One solution. $\tan A'D = \cos A' \div \cot c'$. $A'D < 90°$. $\log \tan A'D = \log \cos A' - \log \cot c' + 10 = 9.329198$. $\therefore A'D = 12° 2' 47''$. $\cot A'B'D = \cos c' \div \cot A'$. $A'B'D < 90°$. $\log \cot A'B'D = \log \cos c' - \log \cot A' + 10 = 10.347007$. $\therefore A'B'D = 24° 13' 1''$. *Check.* $\sin p = \tan A'D \cot A'B'D$, or $\log \sin p = \log \tan A'D + \log \cot A'B'D - 10$.

$\sin C' = \sin p \div \sin a'$. $C' > 90°$. $\log \sin C' = \log \sin p - \log \sin a' + 10 = 9.784174$. $\therefore C' = 142° 31' 40''$. $\cos DC' = \cos a' \div \cos p$. $DC' > 90°$. $\log \cos DC' = \log \cos a'$

$-\log\ \cos p + 10 = 9.851903$. $\therefore\ \mathrm{DC'} = 135°\ 19'\ 15''$. $\cos \mathrm{DB'C'} = \tan p\ \cot a'$. $\mathrm{DB'C'} > 90°$. $\log\ \cos \mathrm{DB'C'} = \log\ \tan p + \log\ \cot a' - 10 = 9.636076$. $\therefore\ \mathrm{DB'C'} = 115°\ 37'\ 55''$. *Check.* $\cos \mathrm{DB'C'} = \cos \mathrm{DC'} \sin \mathrm{C'}$, or $\log \cos \mathrm{DB'C'} = \log\ \cos \mathrm{DC'} + \log\ \sin \mathrm{C'} - 10$. $b' = \mathrm{A'D} + \mathrm{DC'} = 147°\ 22'\ 2''$. $\mathrm{B'} = \mathrm{A'B'D} + \mathrm{DB'C'} = 139°\ 50'\ 56''$. $\mathrm{B} = 180° - b' = 32°\ 37'\ 58''$, $b = 180° - \mathrm{B'} = 40°\ 9'\ 4''$, $c = 180° - \mathrm{C'} = 37°\ 28'\ 20''$. There will be two solutions of the polar, when a' is intermediate in value between p and both c' and $180° - c'$; none when a' lies beyond these. In the original triangle, there will be two solutions when A is intermediate in value between $180° - p$, and both C and $180° - \mathrm{C}$; none when A lies beyond these.

5. $\mathrm{A} = 129°\ 5'\ 28''$, $\mathrm{B} = 142°\ 12'\ 42''$, $\mathrm{C} = 105°\ 8'\ 10''$. $a' = 180° - \mathrm{A} = 50°\ 54'\ 32''$, $b' = 180° - \mathrm{B} = 37°\ 47'\ 18''$, $c' = 180° - \mathrm{C} = 74°\ 51'\ 50''$. $\tan \frac{1}{2}(s + s') : \tan \frac{1}{2}(a' + b') :: \tan \frac{1}{2}(a' - b') : \tan \frac{1}{2}(s - s')$. $s > s'$. $\log \tan \frac{1}{2}(s - s') = a.\ c.\ \log\ \tan \frac{1}{2}(s + s') + \log\ \tan \frac{1}{2}(a' + b') + \log\ \tan \frac{1}{2}(a' - b') - 10 = 9.166914$. $\therefore\ \frac{1}{2}(s - s') = 8°\ 21'\ 18''$. $s = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 45°\ 47'\ 13''$. $s' = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 29°\ 4'\ 37''$. $\cos \mathrm{B'} = \tan s' \cot a'$. $\mathrm{B'} < 90°$. $\log\ \cos \mathrm{B'} = \log\ \tan s + \log\ \cot a' - 10 = 9.921712$. $\therefore\ \mathrm{B'} = 33°\ 22'\ 45''$. $\sin \mathrm{B'C'D} = \sin s \div \sin a'$. $\mathrm{B'C'D'} < 90°$. $\log\ \sin \mathrm{B'C'D} = \log\ \sin s - \log\ \sin a' + 10 = 9.965427$. $\therefore\ \mathrm{B'C'D} = 67°\ 26'\ 24''$. $\cos \mathrm{A'} = \tan s' \cot b'$. $\mathrm{A'} < 90°$. $\log\ \cos \mathrm{A'} = \log\ \tan s' + \log\ \cot b' - 10 = 9.855628$. $\therefore\ \mathrm{A'} = 44°\ 10'\ 40''$. $\sin \mathrm{A'C'D} = \sin s' \div \sin b'$. $\mathrm{A'C'D} < 90°$. $\log\ \sin \mathrm{A'C'D} = \log\ \sin s' - \log\ \sin b' + 10 = 9.899341$. $\therefore\ \mathrm{A'C'D} = 52°\ 27'\ 42''$. $\mathrm{C'} = \mathrm{B'C'D} + \mathrm{A'C'D} = 119°\ 55'\ 6''$. $a = 180° - \mathrm{A'} = 135°\ 49'\ 20''$, $b = 180° - \mathrm{B'} = 146°\ 37'\ 15''$, $c = 180° - \mathrm{C'} = 60°\ 4'\ 54''$.

6. $a = 68°\ 46'\ 2''$, $b = 37°\ 10'$, $\mathrm{C} = 43°\ 37'\ 38''$. $\sin p = \sin b\ \sin \mathrm{C}$. $p < 90°$. $\log\ \sin p = \log\ \sin b + \log\ \sin \mathrm{C} - 10 = 9.619947$. $\therefore\ p = 24°\ 38'\ 2''$. $\tan \mathrm{CD} = \cos \mathrm{C} \div \cot b$. $\mathrm{CD} < 90°$. $\log\ \tan \mathrm{CD} = \log\ \cos \mathrm{C} - \log\ \cot b + 10 =$

9.739398. $\therefore\ CD = 28^\circ\ 45'\ 25''$. $\cot CAD = \cos b \div \cot C$. $CAD < 90^\circ$. $\log \cot CAD = \log \cos b - \log \cot C + 10 =$ 9.880550. $\therefore\ CAD = 52^\circ\ 46'\ 55''$. *Check.* $\sin p = \tan CD \cot CAD$, or $\log \sin p = \log \tan CD + \log \cot CAD - 10$. $DB = a - CD = 40^\circ\ 0'\ 37''$. $\cos c = \cos p \cos DB$. $c < 90$. $\log \cos c = \log \cos p + \log \cos DB - 10 = 9.842748$. $\therefore\ c = 45^\circ\ 52'\ 31''$. $\cot B = \sin DB \div \tan p$. $B < 90^\circ$. $\log \cot B = \log \sin DB - \log \tan p + 10 =$ 10.146772. $\therefore\ B = 35^\circ\ 29'\ 51''$. $\cot BAD = \sin p \div \tan DB$. $BAD < 90^\circ$. $\log \cot BAD = \log \sin p - \log \tan DB + 10 = 9.695976$. *Check.* $\cos c = \cot B \cot BAD$, or $\log \cos c = \log \cot B + \log \cot BAD - 10$. $A = CAD + BAD = 112^\circ\ 22'\ 29''$.

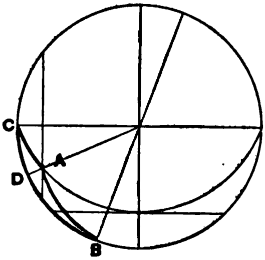

Fig. 29 a.

7. $a = 40^\circ\ 16'$, $b = 47^\circ\ 44'$, $A = 52^\circ\ 34'$. $\sin p = \sin b \sin A$. $p < 90^\circ$. $\log \sin p = \log \sin b + \log \sin A - 10 =$ 9.769099. $\therefore\ p = 35^\circ\ 59'\ 19''$. Two solutions. $\tan AD = \cos A \div \cot b$. $AD < 90^\circ$. $\log \tan AD = \log \cos A - \log \cot b + 10$ $= 9.825288$. $\therefore\ AD = 33^\circ\ 46'\ 25''$. $\cot ACD = \cos b \div \cot A$. $ACD < 90^\circ$. $\log \cot ACD = \log \cos b - \log \cot A + 10 = 9.943811$. $\therefore\ ACD = 48^\circ\ 41'\ 46''$. *Check.* $\sin p = \tan AD \cot ACD$, or $\log \sin p = \log \tan AD + \log \cot ACD - 10$.

$\sin B = \sin p \div \sin a$. $B < 90^\circ$, $> 90^\circ$. $\log \sin B = \log \sin p - \log \sin a + 10 = 9.958634$. $\therefore\ B = 65^\circ\ 23'\ 16''$, and $114^\circ\ 36'\ 44''$. $\cos BCD = \tan p \cot a$. $BCD < 90^\circ$. $\log \cos BCD = \log \tan p + \log \cot a - 10 = 9.933164$. $\therefore\ BCD = 30^\circ\ 58'\ 42''$. $\cos BD = \cos a \div \cos p$. $BD < 90^\circ$. $\log \cos BD = \log \cos a - \log \cos p + 10 = 9.974580$. $\therefore\ BD = 19^\circ\ 25'\ 53''$. *Check.* $\cos BCD$

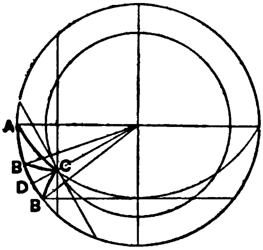

Fig. 30 a.

$= \cos BD \sin B$, or $\log \cos BCD = \log \cos BD + \log \sin B - 10$. $C = ACD \pm BCD = 79° 40' 28'$, and $17° 43' 4''$. $c = AD \pm BD = 53° 12' 18''$, and $14° 20' 32''$. There is but one solution when a is intermediate between p and either b or $180° - b$; none when a is beyond these.

8. $a = 62° 38'$, $b = 10° 13' 19''$, $C = 150° 24' 12''$. $\sin p = \sin b \sin C$. $p < 90°$. $\log \sin p = \log \sin b + \log \sin C - 10 = 8.942737$. $\therefore p = 174° 58' 18''$. $\tan CD = \cos C \div \cot b$. $CD > 90°$. $\log \tan CD = \log \cos C - \log \cot b + 10 = 9.195334$. $\therefore CD = 171° 5' 20''$. $\cot CAD = \cos b \div \cot C$. $CAD > 90°$. $\log \cot CAD = \log \cos b - \log \cot C + 10 = 9.747402$. $\therefore CAD = 119° 12' 17''$. *Check.* $\sin p = \tan CD \cot CAD$, or $\log \sin p = \log \tan CD + \log \cot CAD - 10$. $DB = CD - a = 108° 27' 20''$. $\cos c = \cos p \cos BD$. $c < 90°$. $\log \cos c = \log \cos p + \log \cos BD - 10 = 9.498793$. $\therefore c = 71° 37' 5''$. $\cot ABD = \sin BD \div \tan p$. $ABD > 90°$. $\log \cot ABD = \log \sin BD - \log \tan p + 10 = 11.032651$. $\therefore B = 5° 17' 58''$. $\cot BAD = \sin p \div \tan BD$. $BAD > 90°$. $\log \cot BAD = \log \sin p - \log \tan BD + 10 = 8.466136$. $\therefore BAD = 91° 40' 32''$. *Check.* $\cos c = \cot ABD \cot BAD$, or $\log \cos c = \log \cot ABD + \log \cot BAD - 10$. $A = CAD - BAD = 27° 31' 45''$.

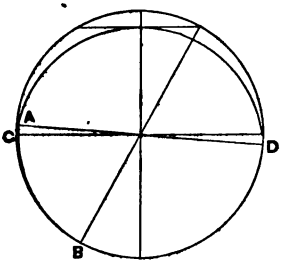

Fig. 31 *a*.

9. For projection, see Fig. 40. $a = 56° 40'$, $b = 83° 13'$, $c = 114° 30'$. $\tan \frac{1}{2}(s + s') : \tan \frac{1}{2}(a + b) :: \tan \frac{1}{2}(a - b) : \tan \frac{1}{2}(s - s')$. $s < s'$. $\log \tan \frac{1}{2}(s - s') = a.\ c.\ \log \tan \frac{1}{2}(s + s') + \log \tan \frac{1}{2}(a + b) + \log \tan \frac{1}{2}(a - b) - 10 = 9.618703$. $\therefore \frac{1}{2}(s - s') = 22° 34' 9''$. $s = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 34° 40' 51''$. $s' = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 79° 49' 9''$. $\cos B = \tan s \cot a$. $B < 90°$. $\log \cos B = \log \tan s + \log \cot a - 10 = 9.658102$. $\therefore B = 62° 55' 44''$.

$\sin BCD = \sin s \div \sin a$. $BCD < 90°$. $\log \sin BCD = \log \sin s - \log \sin a + 10 = 9.833176$. $\therefore BCD = 42° \ 55' \ 32''$. $\cos A = \tan s' \cot b$. $A < 90°$. $\log \cos A = \log \tan s' + \log \cot b - 10 = 9.821091$. $\therefore A = 48° \ 31' \ 16''$. $\sin ACD = \sin s' \div \sin b$. $ACD < 90°$. $\log \sin ACD = \log \sin s' - \log \sin b + 10 = 9.996159$. $\therefore ACD = 82° \ 23' \ 24''$. $C = ACD + BCD = 125° \ 18' \ 56''$.

10. $A = 50° \ 12'$, $B = 58° \ 8'$, $a = 62° \ 42'$. $\sin p = \sin a \sin B$. $p < 90°$. $\log \sin p = \log \sin a + \log \sin B - 10 = 9.877765$. $\therefore p = 48° \ 59' \ 52''$. Two solutions. $\tan BD = \cos B \div \cot a$. $BD < 90°$. $\log \tan BD = \log \cos B - \log \cot a + 10 = 10.009822$. $\therefore BD = 45° \ 38' \ 52''$. $\cot BCD = \cos a \div \cot B$. $BCD < 90°$. $\log \cot BCD = \log \cos a - \log \cot B + 10 = 9.867943$. $\therefore BCD = 53° \ 34' \ 48''$. *Check.* $\sin p = \tan BD \cot BCD$, or $\log \sin p = \log \tan BD + \log \cot BCD - 10$. $\sin b = \sin p \div \sin A$. $b < 90°, > 90°$. $\log \sin b = \log \sin p - \log \sin A + 10 = 9.992243$. $\therefore b = 79° \ 12' \ 10''$, and $100° \ 47' \ 50''$. $\sin ACD = \cos A \div \cos p$. $ACD < 90°, > 90°$. $\log \sin ACD = \log \cos A - \log \cos p + 10 = 9.989292$. $\therefore ACD = 77° \ 19' \ 43''$, and $102° \ 40' \ 17''$. $\sin AD = \tan p \cot A$. $AD < 90°, > 90°$. $\log \sin AD = \log \tan p + \log \cot A - 10 = 9.981535$. $\therefore AD = 73° \ 24' \ 37''$, and $106° \ 35' \ 23''$.

Check. $\sin AD = \sin b \sin ACD$, or $\log \sin AD = \log \sin b + \log \sin ACD - 10$. $C = BCD + ACD = 130° \ 54' \ 31''$, and $156° \ 15' \ 5''$. $c = BD + AD = 119° \ 3' \ 29''$, and $152° \ 14' \ 15''$. There is but one solution, when A is intermediate between p and either B or $180° - B$; none when A is beyond these.

11, 12, and 13 are impossible.

14. $a = 115° \ 20' \ 10''$, $b = 57° \ 30' \ 6''$, $A = 126° \ 37' \ 30''$. $\sin p = \sin b \sin A$. $p > 90°$. $\log \sin p = \log \sin b + \log \sin A - 10 = 9.830513$. $\therefore p = 137° \ 23' \ 58''$. One solution. $\tan AD = \cos A \div \cot b$. $AD > 90°$. $\log \tan AD = \log \cos A - \log \cot b + 10 = 9.971506$. $\therefore AD = 136° \ 52' \ 42''$.

$\cot ACD = \cos b \div \cot A$. $ACD > 90^\circ$. $\log \cot ACD = \log \cos b - \log \cot A + 10 = 9.859007$. $\therefore ACD = 125^\circ\ 51'\ 31''$. *Check.* $\sin p = \tan AD \cot ACD$, or $\log \sin p = \log \tan AD + \log \cot ACD - 10$. $\sin CBD = \sin p \div \sin a$. $CBD > 90^\circ$. $\log \sin CBD = \log \sin p - \log \sin a + 10 = 9.874434$. $\therefore B = 48^\circ\ 29'\ 48''$. $\cos BD = \cos a \div \cos p$. $BD < 90^\circ$. $\log \cos BD = \log \cos a - \log \cos p + 10 = 9.764439$. $\therefore BD = 54^\circ\ 27'\ 16''$. $\cos BCD = \tan p \cot a$. $BCD < 90^\circ$. $\log \cos BCD = \log \tan p + \log \cot a - 10 = 9.638874$. $\therefore BCD = 64^\circ\ 11'\ 25''$. *Check.* $\cos BCD = \cos BD \sin CBD$, or $\log \cos BCD = \log \cos BD + \log \sin CBD - 10$. $C = ACD - BCD = 61^\circ\ 40'\ 6''$. $c = AD - BD = 82^\circ\ 25'\ 26''$.

15. $A = 109^\circ\ 55'\ 42''$, $B = 116^\circ\ 38'\ 33''$, $C = 120^\circ\ 43'\ 37''$. $a' = 180^\circ - A = 70^\circ\ 4'\ 18''$, $b' = 180^\circ - B = 63^\circ\ 21'\ 27''$, $c' = 180^\circ - C = 59^\circ\ 16'\ 23''$. $\tan \frac{1}{2}(s + s') : \tan \frac{1}{2}(b' + c') :: \tan \frac{1}{2}(b' - c') : \tan \frac{1}{2}(s - s')$. $s > s'$. $\log \tan \frac{1}{2}(s - s') = a.\ c.\ \log \tan \frac{1}{2}(s + s') + \log \tan \frac{1}{2}(b' + c') + \log \tan \frac{1}{2}(b' - c') - 10 = 8.968264$. $\therefore \frac{1}{2}(s - s') = 5^\circ\ 18'\ 38''$. $s = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 40^\circ\ 20'\ 47''$. $s' = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 29^\circ\ 43'\ 31''$. $\cos B' = \tan s' \cot c'$. $B' < 90^\circ$. $\log \cos B' = \log \tan s' + \log \cot c - 10 = 9.530691$. $\therefore B' = 70^\circ\ 9'\ 38''$. $\sin B'A'D = \sin s' \div \sin c'$. $B'A'D < 90^\circ$. $\log \sin B'A'D = \log \sin s' - \log \sin c' + 10 = 9.761040$. $\therefore B'A'D = 35^\circ\ 13'\ 38''$. $\cos C' = \tan s \cot b'$. $C' < 90^\circ$. $\log \cos C' = \log \tan s + \log \cot b' - 10 = 9.629576$. $\therefore C' = 74^\circ\ 46'\ 32''$. $\sin C'A'D = \sin s \div \sin b'$. $C'A'D < 90^\circ$. $\log \sin C'A'D = \log \sin s - \log \sin b' + 10 = 9.859927$. $\therefore C'A'D = 46^\circ\ 24'\ 42''$. $A' = B'A'D + C'A'D = 81^\circ\ 38'\ 20''$. $a = 180^\circ - A' = 98^\circ\ 21'\ 40''$, $b = 180^\circ - B' = 109^\circ\ 50'\ 22''$, $c = 180^\circ - C' = 115^\circ\ 13'\ 28''$.

16. Impossible.

17. $a = 50^\circ\ 45'\ 20''$, $b = 69^\circ\ 12'\ 40''$, $A = 44^\circ\ 22'\ 10''$. $\sin p = \sin b \sin A$. $p < 90^\circ$. $\log \sin p = \log \sin b + \log \sin A - 10 = 9.815415$. $\therefore p = 40^\circ\ 49'\ 31''$. Two solutions.

$\tan AD = \cos A \div \cot b$. $AD < 90°$. $\log \tan AD = \log \cos A - \log \cot b + 10 = 10.274837$. $\therefore AD = 62° \; 1' \; 40''$. $\cot ACD = \cos b \div \cot A$. $ACD < 90°$. $\log \cot ACD = \log \cos b - \log \cot A + 10 = 9.540577$. $\therefore ACD = 70° \; 51' \; 11''$. *Check.* $\sin p = \tan AD \cot ACD$, or $\log \sin p = \log \tan AD + \log \cot ACD - 10$. $\sin B = \sin p \div \sin a$. $B < 90°, > 90°$. $\log \sin B = \log \sin p - \log \sin a + 10 = 9.926419$. $\therefore B = 57° \; 34' \; 51''$, and $122° \; 25' \; 9''$. $\cos BD = \cos a \div \cos p$. $BD < 90°$. $\log \cos BD = \log \cos a - \log \cos p + 10 = 9.922223$. $\therefore BD = 33° \; 16' \; 35''$. $\cos BCD = \tan p \cot a$. $BCD < 90°$. $\log \cos BCD = \log \tan p + \log \cot a - 10 = 9.848641$. $\therefore BCD = 45° \; 6' \; 40''$. *Check.* $\cos BCD = \cos BD \sin B$, or $\log \cos BCD = \log \cos BD + \log \sin B - 10$. $c = AD \pm BD = 95° \; 18' \; 15''$, and $28° \; 45' \; 5''$. $C = ACD \pm BCD = 115° \; 57' \; 51''$, and $25° \; 44' \; 31''$.

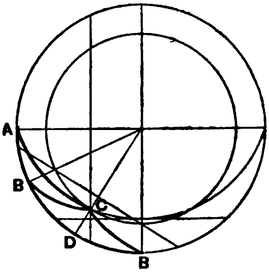

Fig. 32 *a.*

18. $b = 99° \; 40' \; 48''$, $c = 100° \; 49' \; 30''$, $A = 65° \; 33' \; 10''$. $\sin p = \sin b \sin A$. $p < 90°$. $\log \sin p = \log \sin b + \log \sin A - 10 = 9.952977$. $\therefore p = 63° \; 48' \; 57''$. $\tan AD = \cos A \div \cot b$. $AD > 90°$. $\log \tan AD = \log \cos A - \log \cot b + 10 = 10.384935$. $\therefore AD = 112° \; 23' \; 58''$. $\cot ACD = \cos b \div \cot A$. $ACD > 90°$. $\log \cot ACD = \log \cos b - \log \cot A + 10 = 9.568042$. $\therefore ACD = 110° \; 17' \; 51''$. *Check.* $\sin p = \tan AD \cot ACD$, or $\log \sin p = \log \tan AD + \log \cot ACD - 10$. $BD = AD - c = 11° \; 34' \; 28''$. $\cos a = \cos p \cos BD$. $a < 90°$. $\log \cos a = \log \cos p + \log \cos BD - 10 = 9.635771$. $\therefore a = 64° \; 23' \; 14''$. $\cot CBD = \sin BD \div \tan p$. $CBD < 90°$. $\log \cot CBD = \log \sin BD - \log$

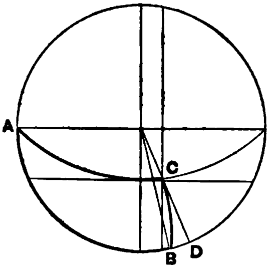

Fig. 33 *a.*

$\tan p + 10 = 8.994135$. $\therefore CBD = 84° 21' 56''$. $\cot BCD = \sin p \div \tan BD$. $BCD < 90°$. $\log \cot BCD = \log \sin p - \log \tan BD + 10 = 10.641636$. $\therefore BCD = 12° 51' 22''$. *Check*. $\cos a = \cot BCD \cot CBD$, or $\log \cos a = \log \cot BCD + \log \cot CBD - 10$. $B = 180° - CBD = 95° 38' 4''$. $C = ACD - BCD = 97° 26' 29''$.

19. $A = 48° 30'$, $B = 125° 20'$, $C = 62° 54'$. $a' = 180° - A = 131° 30'$, $b' = 180° - B = 54° 40'$, $c' = 180° - C = 117° 6'$. $\tan \frac{1}{2}(s+s') : \tan \frac{1}{2}(a'+b') :: \tan \frac{1}{2}(a'-b') : \tan \frac{1}{2}(s-s')$. $s < s'$. $\log \tan \frac{1}{2}(s - s') = a.\ c.\ \log \tan \frac{1}{2}(s + s') + \log \tan \frac{1}{2}(a' + b') + \log \tan \frac{1}{2}(a' - b') - 10 = 10.954459$. $\therefore \frac{1}{2}(s - s') = 83° 39' 47''$. $s = \frac{1}{2}(s + s') - \frac{1}{2}(s - s') = 25° 6' 47''$. $s' = \frac{1}{2}(s + s') + \frac{1}{2}(s - s') = 142° 12' 47''$. $\cos DB'C' = \tan s \cot a'$. $DB'C' > 90°$. $\log \cos DB'C' = \log \tan s + \log \cot a' - 10 = 9.617714$. $\therefore B' = 65° 30' 3''$. $\sin DC'B' = \sin s \div \sin a'$. $DC'B' < 90°$. $\log \sin DC'B' = \log \sin s - \log \sin a' + 10 = 9.753326$. $\therefore DC'B' = 34° 31' 5''$. $\cos A' = \tan s' \cot b'$. $A' > 90°$. $\log \cos A' = \log \tan s' + \log \cot b' - 10 = 9.740071$. $\therefore A' = 123° 20' 30''$. $\sin DC'A' = \sin s' \div \sin b'$. $DC'A' > 90°$. $\log \sin DC'A' = \log \sin s' - \log \sin b' + 10 = 9.875683$. $\therefore DC'A' = 131° 19'$. $C' = DC'A' - DC'B' = 96° 47' 55''$. $a = 180° - A' = 56° 39' 30''$, $b = 180° - B' = 114° 29' 57''$. $c = 180° - C' = 83° 12' 5''$.

20. $C = 54° 15' 3''$, $B = 40° 18' 13''$, $a = 70° 30' 30''$. $c' = 180° - C = 125° 44' 57''$, $b' = 180° - B = 139° 41' 47''$, $A' = 180° - a = 109° 29' 30''$. $\sin p = \sin b' \sin A'$. $p > 90°$. $\log \sin p = \log \sin b' + \log \sin A' - 10 = 9.785164$. $\therefore p = 142° 25' 38''$. $\tan A'D = \cos A' \div \cot b'$. $A'D < 90°$. $\log \tan A'D = \log \cos A' - \log \cot b' + 10 = 9.451800$. $\therefore A'D = 15° 48' 7''$. $\cot A'C'D = \cos b' \div \cot A'$. $A'C'D < 90°$. $\log \cot A'C'D = \log \cos b' - \log \cot A' + 10 = 10.333364$. $\therefore A'C'D = 24° 53' 50''$. *Check*. $\sin p = \tan A'D \cot A'C'D$, or $\log \sin p = \log \tan A'D + \log \cot A'C'D - 10$. $DB' = c' - A'D = 109° 56' 50''$. $\cos a' = \cos p \cos DB'$. $a' < 90°$. $\log \cos a' = \log \cos p + \log \cos DB' - 10 = 9.431994$. $\therefore a' = 74° 18' 45''$. $\cot B'$

$= \sin DB' \div \tan p$. $B' > 90°$. $\log \cot B' = \log \sin DB' - \log \tan p + 10 = 10.087009$. $\therefore B' = 140° \, 42' \, 5''$. $\cot DC'B' = \sin p \div \tan DB'$. $DC'B' > 90°$. $\log \cot DC'B' = \log \sin p - \log \tan DB' + 10 = 9.344984$. $\therefore DC'B' = 102° \, 28' \, 43''$. *Check.* $\cos a' = \cot B' \cot DC'B'$, or $\log \cos a = \log \cot B' + \log \cot DC'B' - 10$. $C' = A'C'D + B'C'D = 127° \, 22' \, 33''$. $A = 180° - a' = 105° \, 41' \, 15''$, $b = 180° - B' = 39° \, 17' \, 55''$, $c = 180° - C' = 52° \, 37' \, 27''$.

21. The perpendicular and side opposite are so nearly equal that no satisfactory projection can be made. $A = 47° \, 54' \, 21''$, $c = 61° \, 4' \, 56''$, $a = 40° \, 31' \, 20''$. $\sin p = \sin c \sin A$. $p < 90°$. $\log \sin p = \log \sin c + \log \sin A - 10 = 9.812594$. $\therefore p = 40° \, 30' \, 20''$. Two solutions. $\tan AD = \cos A \div \cot c$. $AD < 90°$. $\log \tan AD = \log \cos A - \log \cot c + 10 = 10.084021$. $\therefore AD = 50° \, 30' \, 29''$. $\cot ABD = \cos c \div \cot A$. $ABD < 90°$. $\log \cot ABD = \log \cos c - \log \cot A + 10 = 9.728572$. $\therefore ABD = 61° \, 50' \, 29''$. *Check.* $\sin p = \tan AD \cot ABD$, or $\log \sin p = \log \tan AD + \log \cot ABD - 10$. $\sin C = \sin p \div \sin a$. $C > 90°, < 90°$. $\log \sin C = \log \sin p - \log \sin a + 10 = 9.999852$. $\therefore C = 88° \, 30' \, 17''$, and $91° \, 29' \, 43''$. $\cos DC - \cos a \div \cos p$. $DC < 90°$. $\log \cos DC = \log \cos a - \log \cos p + 10 = 9.999892$. $\therefore DC = 1° \, 16' \, 40''$. $\cos DBC = \tan p \cot a$. $DBC < 90°$. $\log \cos DBC = \log \tan p + \log \cot a - 10 = 9.999744$. $\therefore DBC = 1° \, 58'$. *Check.* $\cos DBC = \cos DC \sin C$, or $\log \cos DBC = \log \cos DC + \log \sin C - 10$. $b = AD \pm DC = 51° \, 47' \, 9''$, and $49° \, 13' \, 49''$. $B = ABD \pm DBC = 63° \, 48' \, 29''$, and $59° \, 52' \, 29''$.

22. Impossible.

AREA OF SPHERICAL TRIANGLES.

Page 108, 155.

2. $K = S - \pi = \frac{420}{180}\pi - \pi = \frac{4}{3}\pi.$ $\frac{4}{3}\pi \times 10^2 = 418.88.$

3. $K = S - \pi = \frac{320}{180}\pi - \pi = \frac{7}{9}\pi.$ $\frac{7}{9}\pi \times 3^2 = 21.9912.$

Page 110, 156.

2. $a = 70° 14' 20''$, $b = 49° 24' 10''$, $c = 38° 46' 10''$. $\tan \frac{1}{4} K = \sqrt{\tan \frac{1}{4} s \tan (\frac{1}{4} s - \frac{1}{2} a) \tan (\frac{1}{4} s - \frac{1}{2} b) \tan (\frac{1}{4} s - \frac{1}{2} c)}$. $\log \tan \frac{1}{4} K = \frac{1}{2} \{\log \tan \frac{1}{4} s + \log \tan (\frac{1}{4} s - \frac{1}{2} a) + \log \tan (\frac{1}{4} s - \frac{1}{2} b) + \log \tan (\frac{1}{4} s - \frac{1}{2} c)\} - 10 = 8.901648$. $\therefore \frac{1}{4} K = 4° 33' 32''$, $K = 18° 14' 8'' = 18°.2356$. Area $= \frac{18.2356}{90} \times \frac{\pi}{2} \times 4^2 = 5.1-$.

PRACTICAL APPLICATIONS.

Page 110, Prob. 1.

1. Let b = co-latitude of Alexandria $= 58° 47'$, c = co-latitude of Berlin $= 37° 28' 47''$, A = difference in longitude $= 16° 31' 8''$. $\tan \varphi = \tan b \cos A$. $\log \tan \varphi = \log \tan b + \log \cos A - 10 = 10.199208$. $\therefore \varphi = 57° 42' 10''$. $\cos a = \frac{\cos (c - \varphi) \cos b}{\cos \varphi}$. $\log \cos a = \log \cos (c - \varphi) + \log \cos b + a.\ c.\ \log \cos \varphi - 10 = 9.959134$. $\therefore a = 24° 28' 4'' = 24°.4678-$. In miles, $a = \frac{3962 \times \pi}{180°} \times 24°.4678- = 1692$.

2. Let c = co-latitude of Valparaiso $= 56° 58'$, A = difference in longitude $= 48° 17'$, a = length of arc $= \frac{3840 \times 180°}{3962 \times \pi} = 55°.53154 = 55° 31' 54''$. $\tan \varphi = \tan c \cos A$. $\log \tan \varphi = \log \tan c + \log \cos A - 10 = 10.010044$. $\therefore \varphi = 45° 39' 45''$.

$\cos\varphi' = \dfrac{\cos\varphi\cos a}{\cos c}$. $\log\cos\varphi' = \log\cos\varphi + \log\cos a + a.c.$ $\log\cos c - 10 = 9.860684$. $\therefore\ \varphi' = 43°\ 28'\ 59''$. $b = \varphi + \varphi' = 89°\ 8'\ 44''$. $\therefore$ the latitude is $90° - b = 51'\ 16''$ S.

3. Let c = co-latitude of Rio Janeiro $= 67°\ 6'$, b = co-latitude of her final position $= 140°$, a = length of arc $= \dfrac{5624.4 \times 180°}{3962 \times \pi} = 81°.3364 = 81°\ 20'\ 11''$. $\sin\frac{1}{2}\text{A} =$

$$\sqrt{\frac{\sin(\frac{1}{2}s - b)\sin(\frac{1}{2}s - c)}{\sin b\sin c}}.$$

$\log\sin\frac{1}{2}\text{A} = \frac{1}{2}\{\log\sin(\frac{1}{2}s - b) + \log\sin(\frac{1}{2}s - c) + a.\,c.\log\sin b + a.\,c.\log\sin c\} = 9.541564$. $\therefore\ \frac{1}{2}\text{A} = 20°\ 21'\ 51''$, $\text{A} = 40°\ 43'\ 52''$. Longitude $= 42°\ 45' - 40°\ 43'\ 52'' = 2°\ 1'\ 18''$ W.

Page 112, Prob. 2.

2. Let a = co-latitude $= 49°\ 39'$, b = co-declination of sun $= 93°\ 20$, c = co-altitude $= 53°\ 48$. $\sin\frac{1}{2}\text{C} =$

$$\sqrt{\frac{\sin(\frac{1}{2}s - a)\sin(\frac{1}{2}s - b)}{\sin a\sin b}}.$$

$\log\sin\frac{1}{2}\text{C} = \frac{1}{2}\{\log\sin(\frac{1}{2}s - a) + \log\sin(\frac{1}{2}s - b) + a.\,c.\log\sin a + a.\,c.\log\sin b\} = 9.470055$. $\therefore\ \frac{1}{2}\text{C} = 17°\ 10'\ 1''$, $\text{C} = 34°\ 20'\ 2''$, which reduced to time gives 2*h*. 17*m*. 20*s*. before noon, or 9*h*. 42*m*. 40*s*. A.M.

3. $a = 111°\ 2'$, $b = 71°\ 28'$, $c = 49°\ 52'$. $\log\sin\frac{1}{2}\text{C} = 9.426750$. $\therefore\ \frac{1}{2}\text{C} = 15°\ 29'\ 40''$, $\text{C} = 30°\ 59'\ 20''$, which reduced to time gives 2*h*. 3*m*. 57*s*. P.M.

Page 113, Prob. 3.

2. $\sin\text{CL} = \tan\text{lat}\tan\text{dec}$. $\log\sin\text{CL} = \log\tan\text{lat} + \log\tan\text{dec} - 10 = 9.825896$. $\therefore\ \text{CL} = 42°\ 2'\ 45''$, which reduced to time gives 2*h*. 48*m*. 11*s*. before 6 o'clock, or 3*h*. 11*m*. 49*s*. A.M.

3. Similarly, $\log\sin\text{CL} = 9.736709$. $\therefore\ \text{CL} = 33°\ 3'\ 5''$, which reduced to time gives 2*h*. 12*m*. 12*s*. after 6 o'clock,

or 8*h*. 12*m*. 12*s*. A.M., as the time of sunrise. From this we see that the day is 2 (12*h*. − 8*h*. 12*m*. 12*s*.), or 7*h*. 35*m*. 36*s*. long.

4. Similarly, log sin CL = 9.595970. ∴ CL = 23° 13′ 49″, which reduced to time gives 1*h*. 32*m*. 55*s*. before 6 o'clock, as the time of sunrise. From this we see that the day is 2 (6*h*. + 1*h*. 32*m*. 55*s*.), or 15*h*. 5*m*. 50*s*. long.

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